An energy principle for hydromagnetic stability problems

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The problem of the stability of static, highly conducting, fully ionized plasmas is investigated by means of an energy principle developed from one introduced by Lundquist. The derivation of the principle and the conditions under which it applies are given. The method is applied to find complete stability criteria for two types of equilibrium situations. The first concerns plasmas which are completely separated from the magnetic field by an interface. The second is the general axisymmetric system.

1. Introduction

The investigation of hydromagnetic systems and their stability is of interest in such varied fields as the study of sunspots, interstellar matter, terrestrial magnetism, auroras and gas discharges. An excellent summary and bibliography of these applications has been given by Elsasser (1955, 1956). The stability of hydromagnetic systems has been extensively investigated in a fundamental series of papers by Chandrasekhar (1952 to 1956).

The present work is concerned with those hydromagnetic equilibria in which the fluid velocity at each point is assumed to vanish. It is divided into two parts. The first is a development of an energy principle, originally stated by Lundquist (1951, 1952), for investigating the stability of such systems. The second part consists of the application of this principle to obtain a number of specific results for such systems.

The 'normal mode' technique is the usual method for the investigation of stability in many systems, mechanical, electrical, etc. It consists of solving the linearized equations of motion for small perturbations about an equilibrium state. The system is said to be unstable if any solution increases indefinitely in time; if no such solution exists, the system is stable.

The energy principle technique, on the other hand, depends upon a variational formulation of the equations of motion. It was first used by Rayleigh (1877) in the calculation of the frequencies of vibrating systems. Its advantage lies in the fact that if one seeks solely to determine stability, and not rates of growth or oscillation frequencies, it is necessary only to discover whether there is any perturbation which decreases the potential energy from its equilibrium value. This makes practical the stability analysis of much more complicated equilibria than the normal mode method.

In § 2 are presented the basic equations for a plasma and the conditions under which they are valid. These equations are then linearized in the Lagrangian representation. In § 3, the energy principle is stated and derived from the normal mode equations for the system. The relation between the energy principle and Rayleigh's principle (Rayleigh 1877) is discussed.

In § 4, some convenient methods for applying the energy principle to general problems are described. In § 5, the problem of the stability of a fluid in which the magnetic field is zero and which is surrounded by a vacuum magnetic field is solved.

Section 6 treats the stability of a general axisymmetric system. The problem is reduced essentially to the solution of an ordinary second-order eigenvalue equation. In certain limiting situations the problem is solved completely.

2. Basic considerations

Consider a plasma of electrons and of one kind of positive ion which is governed by the following system of equations:

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\operatorname{grad} p + \mathbf{j} \times \mathbf{B} - \rho \operatorname{grad} \phi, \tag{2·1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{2·2}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \tag{2·3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\operatorname{grad}\right)(p\rho^{-\gamma}) = 0, \tag{2·4}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2·5}$$

$$\operatorname{curl} \mathbf{B} = \mathbf{j}, \tag{2·6}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{2·7}$$

Let $\mathbf{E}$ be the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{j}$ the electric current density, ρ the mass density, M the ion mass, p the pressure, ϕ the external potential energy per unit mass, γ the ratio of specific heats, e the magnitude of the electronic charge and $\mathbf{v}$ the fluid velocity. The equations are written in rationalized Gaussian units with $c = 1$.

The above equations apply if the following conditions are satisfied: (i) Quadratic terms in $\mathbf{v}$ and $\mathbf{j}$ are negligible. Physically, this is equivalent to the requirement that the macroscopic speed v is small compared to sound speed $c_s = (\gamma p/\rho)^{\frac{1}{2}}$ or to hydromagnetic speed $c_k = B/\sqrt{\rho}$. (ii) The system is locally electrically quasi-neutral. This occurs if the Debye shielding distance $\lambda_D = (kT_e/ne^2)^{\frac{1}{2}}$ is small compared to every characteristic dimension L of the system. (iii) The ratio of the electron mass, m to the ion mass, M is negligible compared to unity. (iv) The matter stress tensor is isotropic. This occurs if there are many collisions during a characteristic time, t_c. The effect of relaxing the requirement of isotropy of the stress tensor is considered in § 3. (v) The displacement current is negligible. This holds if c_k is small compared to the speed of light. (vi) Heat flow by conduction, along the lines of force as well as across the lines, is negligible. This implies the adiabatic law (2·4). It is shown in § 3 how this law must be modified if conditions (iv) and (vii) are not satisfied. (vii) Ohm's law in the form of equation (2·3) is valid.

Spitzer (1956) gives the complete generalized Ohm's law which may be written in the form

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{M}{e} \operatorname{grad} \phi - \eta \mathbf{j} - \frac{m}{ne^2} \frac{\partial \mathbf{j}}{\partial t} - \frac{1}{ne} \operatorname{grad} p_i - \frac{M}{e} \frac{\partial \mathbf{v}}{\partial t} = 0.$$

The electron inertia term $(m/ne^2)\,\partial\mathbf{j}/\partial t$ is negligible when $(t_c)^{-1}$ is small compared to the electron plasma frequency $\omega_p = (ne^2/m)^{\frac{1}{2}}$. The ion inertia term $(M/e)\,\partial\mathbf{v}/\partial t$ is negligible when $(t_c)^{-1}$ is small compared to the ion Larmor frequency eB/M. The electrical resistance term $\eta\mathbf{j}$ is negligible when the time characteristic of relative diffusion of matter and magnetic flux is long compared to t_c. The terms involving $\operatorname{grad}\phi$ and $\operatorname{grad} p_i$ are negligible when $a_i c_s/Lv \ll 1$, where a_i is the ion Larmor radius. Spitzer has pointed out that this criterion is not satisfied in general for fully ionized plasmas. In particular, for equilibrium states in which v is zero, the criterion fails. The effect of keeping these terms is discussed in § 3 where it is shown that the stability criteria are not affected by their inclusion.

The set of equations above implies relations between quantities on adjacent sides of an interface, either interior to the fluid or between fluid and vacuum. Denote by $\mathbf{n}$ the unit normal to the interface, by $\mathbf{K}$ the surface current density, and by $\langle X\rangle$ the increment in any quantity X across the boundary in the direction $\mathbf{n}$. For a fluid-fluid interface the relations are

$$\langle p+\tfrac{1}{2}B^2\rangle = 0, \tag{2·8}$$

$$\mathbf{n}.\langle\mathbf{v}\rangle = 0, \tag{2·9}$$

$$\mathbf{n}\times\langle\mathbf{E}\rangle = \mathbf{n}.\mathbf{v}\langle\mathbf{B}\rangle, \tag{2·10}$$

$$\mathbf{n}.\langle\mathbf{B}\rangle = 0, \tag{2·11}$$

$$\mathbf{n}\times\langle\mathbf{B}\rangle = \mathbf{K}. \tag{2·12}$$

For a fluid-vacuum interface equation (2·9) is meaningless, but the remaining relations apply with $\mathbf{v}$ taken to be the fluid velocity.

The region of interest can often be considered surrounded by a rigid, perfectly conducting wall. At such a boundary the appropriate conditions are

$$\mathbf{n}\times\mathbf{E} = 0, \tag{2·13}$$

$$\mathbf{n}.\partial\mathbf{B}/\partial t = 0, \tag{2·14}$$

$$\mathbf{n}.\mathbf{v} = 0. \tag{2·15}$$

A further condition which must be satisfied at any interface carrying a sheet current, but no sheet mass, is that the lines of force of the magnetic field lie in the interface. This arises from the fact that refraction of the lines of force would give rise to infinite accelerations in the surface due to the unbalanced tangential forces. Throughout this paper, only surfaces of discontinuity will be considered at which the condition $\mathbf{n}.\mathbf{B} = 0$ is satisfied. This is the boundary condition of interest, for example, for a confined plasma in which gravitational effects are negligible.

It can be shown that the system of equations above possess an energy integral

$$U = \int d\tau\left(\tfrac{1}{2}\rho\,|\,\mathbf{v}\,|^2+\tfrac{1}{2}\,|\,\mathbf{B}\,|^2+\frac{p}{\gamma-1}+\rho\phi\right) = \text{constant}, \tag{2·16}$$

where the integration is extended over the whole domain, fluid and vacuum.

It is convenient in later exhibiting the energy principle for the linearized form of the above equations to adopt a Lagrangian description of the fluid motion. Accordingly, all quantities are now considered to be functions of $\mathbf{r}_0$, the initial location of a fluid element, and of t, the time. Let the displacement vector $\boldsymbol{\xi}(\mathbf{r}_0, t)$ be determined by

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\xi}, \tag{2·17}$$

where $\mathbf{r}$ is the location of the fluid element at time t. Clearly $\boldsymbol{\xi}(\mathbf{r}_0, 0)$ is zero. Define grad_0 to be the gradient operator with respect to $\mathbf{r}_0$. The usual chain rule of differentiation yields

$$\text{grad} = \text{grad}\,\mathbf{r}_0 \,.\, \text{grad}_0. \tag{2·18}$$

To first order in $\boldsymbol{\xi}$ equation (2·18) becomes

$$\text{grad} = \text{grad}_0 - (\text{grad}_0 \boldsymbol{\xi}) \,.\, \text{grad}_0. \tag{2·19}$$

Consider systems which are passing through a configuration of static equilibrium at time zero. The equilibrium equations are

$$\text{grad}_0 p_0 - \mathbf{j}_0 \times \mathbf{B}_0 + \rho_0 \,\text{grad}_0 \phi_0 = 0, \tag{2·20}$$

$$\text{curl}\,\mathbf{B}_0 = \mathbf{j}_0, \tag{2·21}$$

$$\text{div}_0 \mathbf{B}_0 = 0. \tag{2·22}$$

The equations determining the various perturbed field quantities at $\mathbf{r}$ to first order in $\boldsymbol{\xi}$ are determined by linearizing (2·1) to (2·6). There results on combining (2·3) and (2·5) and integrating in time,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{Q} + \boldsymbol{\xi} \,.\, \text{grad}_0 \mathbf{B}_0, \tag{2·23}$$

where

$$\mathbf{Q} = \text{curl}_0 (\boldsymbol{\xi} \times \mathbf{B}_0). \tag{2·24}$$

Equations (2·6), (2·2), (2·4) and a Taylor expansion of the external potential yield, respectively,

$$\mathbf{j} = \mathbf{j}_0 - [(\text{grad}_0 \boldsymbol{\xi}) \,.\, \text{grad}_0] \times \mathbf{B}_0 + \text{curl}_0 \mathbf{Q} + \text{curl}_0 [(\boldsymbol{\xi} \,.\, \text{grad}_0)\, \mathbf{B}_0], \tag{2·25}$$

$$\rho = \rho_0 - \rho_0 \,\text{div}_0 \boldsymbol{\xi}, \tag{2·26}$$

$$p = p_0 - \gamma p_0 \,\text{div}_0 \boldsymbol{\xi}, \tag{2·27}$$

$$\phi = \phi_0 + \boldsymbol{\xi} \,.\, \text{grad}_0 \phi_0. \tag{2·28}$$

The above equations are the first-order Lagrangian counterparts of (2·2) to (2·6). Note that they involve $\boldsymbol{\xi}$ but not $\dot{\boldsymbol{\xi}}$, where a dot indicates differentiation with respect to time. It can be shown that this property of depending on $\boldsymbol{\xi}$ but not $\dot{\boldsymbol{\xi}}$ holds for the expression of grad, $\mathbf{B}$, $\mathbf{j}$, ρ, p and ϕ to all higher orders in $\boldsymbol{\xi}$. Finally, the equation of motion (2·1) takes the form

$$\rho_0 \ddot{\boldsymbol{\xi}} = \mathbf{F}\{\boldsymbol{\xi}\}, \tag{2·29}$$

where

$$\mathbf{F}\{\boldsymbol{\xi}\} = \text{grad}_0 [\gamma p_0 \,\text{div}_0 \boldsymbol{\xi} + (\boldsymbol{\xi} \,.\, \text{grad}_0)\, p_0] + \mathbf{j}_0 \times \mathbf{Q} - \mathbf{B}_0 \times \text{curl}_0 \mathbf{Q} + [\text{div}_0 (\rho_0 \boldsymbol{\xi})]\, \text{grad}_0 \phi_0. \tag{2·30}$$

Note that $\mathbf{F}$ also depends only on $\boldsymbol{\xi}$ and not on $\dot{\boldsymbol{\xi}}$.

Note that (2·29) with appropriate initial and boundary conditions determines $\boldsymbol{\xi}$. Equations (2·23) to (2·28) then determines the perturbed field quantities.

The boundary conditions at an interface between a plasma and a vacuum are given by transcribing (2·8) to (2·12) to first order $\boldsymbol{\xi}$. Introduce the first-order vacuum vector potential, $\mathbf{A}$, where

$$\hat{\mathbf{E}} = -\frac{\partial \mathbf{A}}{\partial t} + \hat{\mathbf{E}}_0 \quad \text{and} \quad \hat{\mathbf{B}} = \text{curl}\,\mathbf{A} + \hat{\mathbf{B}}_0, \tag{2·31}$$

and vacuum quantities are distinguished when necessary by a circumflex. The gauge has been chosen so that the scalar potential vanishes. Then from (2·8)

$$-\gamma p_0 \operatorname{div}_0 \boldsymbol{\xi} + \mathbf{B}_0 . (\mathbf{Q} + [\boldsymbol{\xi} . \operatorname{grad}_0] \mathbf{B}_0) = \hat{\mathbf{B}}_0 . (\operatorname{curl} \mathbf{A} + [\boldsymbol{\xi} . \operatorname{grad}] \hat{\mathbf{B}}_0). \tag{2·32}$$

It follows from (2·11), (2·10) and (2·3) that

$$\mathbf{n}_0 \times \mathbf{A} = -(\mathbf{n}_0 . \boldsymbol{\xi}) \hat{\mathbf{B}}_0. \tag{2·33}$$

Of course, $\mathbf{A}$ must satisfy the equation

$$\operatorname{curl} (\operatorname{curl} \mathbf{A}) = 0 \tag{2·34}$$

in the vacuum.

Equations (2·33) and (2·34) serve to determine $\operatorname{curl}_0 \mathbf{A}$ in terms of $\boldsymbol{\xi}$, so that (2·32) is the only constraint on $\boldsymbol{\xi}$. The linearized counterpart of (2·13) which holds at a rigid, perfectly conducting wall bounding the vacuum is

$$\hat{\mathbf{n}} \times \mathbf{A} = 0. \tag{2·35}$$

At such a wall bounding a fluid, the condition is

$$\mathbf{n} . \boldsymbol{\xi} = 0. \tag{2·36}$$

3. The energy principle

On the basis of § 2, it is possible in principle to follow in time any small motion about an equilibrium state in which the fluid velocity is zero. The central problem of this paper is to determine for a given equilibrium configuration whether such a small motion grows in time. If we confine ourselves just to the question of the determination of the stability of a system and do not inquire into details of the motion, the problem may be reduced to examining the sign of the change in the potential energy as a functional of $\boldsymbol{\xi}$. It will be shown in this section that the system is unstable if, and only if, there exists some displacement $\boldsymbol{\xi}$ which makes this change in energy negative.

The demonstration demands that $\mathbf{F}$ ((2·30)) be a self-adjoint operator. That is, for any two vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfying (2·32)

$$\int \mathrm{d}\tau_0 \boldsymbol{\eta} . \mathbf{F}\{\boldsymbol{\xi}\} = \int \mathrm{d}\tau_0 \boldsymbol{\xi} . \mathbf{F}\{\boldsymbol{\eta}\}. \tag{3·1}$$

The self-adjointness property of F could be proved directly, but will be shown more simply to follow from the existence of an energy integral for the linearized system in which terms in the form of a product of $\dot{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ do not appear. Such an energy integral for the linearized system is guaranteed in the case $\mathbf{v} = 0$ by the energy integral, (2·16), for the exact equations. In fact, the kinetic energy term for the linearized system is just

$$K\{\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\} = \frac{1}{2} \int \mathrm{d}\tau_0 \rho_0 |\dot{\boldsymbol{\xi}}|^2, \tag{3·2}$$

while, when the potential energy terms are expanded in $\boldsymbol{\xi}$, the change in the potential energy is a quadratic form $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ which does not involve $\dot{\boldsymbol{\xi}}$ because of the remark following (2·28). Hence,

$$K\{\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\} + \delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\} \tag{3·3}$$

is constant. One obtains from the equation of motion (2·29)

$$\dot{K} = \int d\tau_0 \dot{\boldsymbol{\xi}} . \mathbf{F}\{\boldsymbol{\xi}\} = -\delta\dot{W}$$
$$= -\delta W\{\dot{\boldsymbol{\xi}}, \boldsymbol{\xi}\} - \delta W\{\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\}. \tag{3·4}$$

Since $\dot{\boldsymbol{\xi}}$ satisfies the same boundary condition as $\boldsymbol{\xi}$, we can choose $\dot{\boldsymbol{\xi}}$ to be equal to any arbitrary displacement $\boldsymbol{\eta}$. By (3·4)

$$\int d\tau_0 \dot{\boldsymbol{\xi}} . \mathbf{F}\{\boldsymbol{\xi}\} = \int d\tau_0 \boldsymbol{\xi} . \mathbf{F}\{\dot{\boldsymbol{\xi}}\} \tag{3·5}$$

and $\mathbf{F}$ is self-adjoint. Further the potential energy is

$$\delta W = -\frac{1}{2} \int d\tau_0 \boldsymbol{\xi} . \mathbf{F}\{\boldsymbol{\xi}\}, \tag{3·6}$$

as seen by replacing $\dot{\boldsymbol{\xi}}$ by $\boldsymbol{\xi}$ itself in (3·4).

Since the time does not appear explicitly in (2·29), one seeks normal mode solutions of the form

$$\boldsymbol{\xi}_n(\mathbf{r}_0, t) = \boldsymbol{\xi}_n(\mathbf{r}_0)\, e^{i\omega_n t}. \tag{3·7}$$

The corresponding eigenvalue equation is

$$-\omega_n^2 \rho_0 \boldsymbol{\xi}_n = \mathbf{F}\{\boldsymbol{\xi}_n\}, \tag{3·8}$$

where $\boldsymbol{\xi}_n$ satisfies the boundary condition (2·32). Since $\mathbf{F}$ is self-adjoint the eigenfunctions $\boldsymbol{\xi}_n$ can be chosen to satisfy the orthonormality condition

$$\frac{1}{2} \int d\tau_0 \rho_0 \boldsymbol{\xi}_n . \boldsymbol{\xi}_m = \delta_{nm}. \tag{3·9}$$

It is physically reasonable to assume that these eigenfunctions form a complete set for any functions which satisfy the boundary condition (2·32). (The unimportant special cases involving degeneracy of eigenfunctions will be consistently ignored.) It further follows from the fact that $\mathbf{F}$ is self-adjoint that ω_n^2 is real and thus the phenomenon of 'overstability' cannot occur.

Any eigenmode with positive ω_n^2 corresponds to a stable oscillation. A negative ω_n^2 corresponds to instability. Thus, in virtue of the assumed completeness property, the necessary and sufficient condition for instability is the existence of a negative ω_n^2.

On physical grounds one expects that if δW can be made negative then the system is unstable and therefore, there exists at least one negative ω_n^2. To show this, let $\boldsymbol{\xi}$ be a displacement which satisfies the boundary condition (2·32) and for which $\delta W < 0$. By the assumed completeness property one can write

$$\boldsymbol{\xi} = \Sigma a_n \boldsymbol{\xi}_n, \tag{3·10}$$

and from (3·6), (3·8) and (3·9)

$$\delta W = -\tfrac{1}{2} \sum_n \sum_m a_n a_m \int d\tau_0 \boldsymbol{\xi}_n . \mathbf{F}\{\boldsymbol{\xi}_m\}$$
$$= \tfrac{1}{2} \sum_n a_n^2 \omega_n^2. \tag{3·11}$$

Thus δW can be made negative if and only if there exists at least one negative ω_n^2. Therefore, the determination of the stability of a system is reduced to an examination of the sign of δW. Since the displacements $\boldsymbol{\xi}$ which may be employed in δW are subject to (2·32), the energy principle as it stands is of limited utility. It is possible to derive an extended energy principle which dispenses with this constraint.

To this end one rewrites δW as the sum of three terms, a volume integral δW_F extended over the fluid domain, a surface integral δW_S extended over the fluid-vacuum interface and a volume integral δW_V extended over the vacuum region. There results from (2·30) and (3·6) after integration by parts, suppression of the subscript zero and use of the condition $\mathbf{n}.\mathbf{B} = 0$,

$$\delta W = \delta W_F - \frac{1}{2}\int \mathrm{d}\sigma\, \mathbf{n}.\boldsymbol{\xi}[\gamma p \operatorname{div}\boldsymbol{\xi} + \boldsymbol{\xi}.\operatorname{grad} p - \mathbf{B}.\mathbf{Q}], \tag{3·12}$$

where

$$\delta W_F = \frac{1}{2}\int \mathrm{d}\tau\{|\mathbf{Q}|^2 - \mathbf{j}.\mathbf{Q}\times\boldsymbol{\xi} + \gamma p(\operatorname{div}\boldsymbol{\xi})^2 + (\operatorname{div}\boldsymbol{\xi})(\boldsymbol{\xi}.\operatorname{grad} p) - (\boldsymbol{\xi}.\operatorname{grad}\phi)\operatorname{div}(\rho\boldsymbol{\xi})\}, \tag{3·13}$$

and the integral is extended, of course, over the initial volume of the fluid. Note that the continuity of the equilibrium value of $(p+\frac{1}{2}|\mathbf{B}|^2)$ across the boundary implies the continuity of $\mathbf{n}\times\operatorname{grad}(p+\frac{1}{2}|\mathbf{B}|^2)$. This allows us with the help of equation (2·32) to rewrite the surface term in (3·12) as

$$\begin{aligned}\delta W - \delta W_F &= \frac{1}{2}\int \mathrm{d}\sigma\, \mathbf{n}.\boldsymbol{\xi}\{-\boldsymbol{\xi}.\operatorname{grad}(p+\tfrac{1}{2}|\mathbf{B}|^2) + \boldsymbol{\xi}.\operatorname{grad}(\tfrac{1}{2}|\hat{\mathbf{B}}|^2) + \hat{\mathbf{B}}.\operatorname{curl}\mathbf{A}\}\\ &= \frac{1}{2}\int \mathrm{d}\sigma\{-(\mathbf{n}.\boldsymbol{\xi})^2\,\mathbf{n}.\operatorname{grad}(p+\tfrac{1}{2}|\mathbf{B}|^2) - (\mathbf{n}.\boldsymbol{\xi})^2\,\hat{\mathbf{n}}.\operatorname{grad}(\tfrac{1}{2}|\hat{\mathbf{B}}|^2)\\ &\qquad - (\hat{\mathbf{n}}.\boldsymbol{\xi})\,\hat{\mathbf{B}}.\operatorname{curl}\mathbf{A}\}.\end{aligned} \tag{3·14}$$

Further, employing (2·33) we obtain

$$\begin{aligned}-\int \mathrm{d}\sigma(\hat{\mathbf{n}}.\boldsymbol{\xi})\,\hat{\mathbf{B}}.\operatorname{curl}\mathbf{A} &= \int \mathrm{d}\sigma\,\hat{\mathbf{n}}\times\mathbf{A}.\operatorname{curl}\mathbf{A}\\ &= \int \mathrm{d}\hat{\tau}\operatorname{div}(\mathbf{A}\times\operatorname{curl}\mathbf{A})\\ &= \int \mathrm{d}\hat{\tau}\{|\operatorname{curl}\mathbf{A}|^2 - \mathbf{A}.\operatorname{curl}(\operatorname{curl}\mathbf{A})\}.\end{aligned} \tag{3·15}$$

Thus, in virtue of (2·34) the final form of δW is

$$\delta W = \delta W_F + \delta W_S + \delta W_V, \tag{3·16}$$

where δW_F is given by (3·13),

$$\delta W_V = \frac{1}{2}\int \mathrm{d}\hat{\tau}\,|\operatorname{curl}\mathbf{A}|^2 \tag{3·17}$$

and

$$\delta W_S = \frac{1}{2}\int \mathrm{d}\sigma\,(\mathbf{n}.\boldsymbol{\xi})^2\,\mathbf{n}.\langle\operatorname{grad}(p+\tfrac{1}{2}|\mathbf{B}|^2)\rangle. \tag{3·18}$$

With this form for δW, (3·16), the energy principle will now be extended to displacements $\boldsymbol{\xi}$ which do not satisfy the constraint equation (2·32). It will be shown that if there exist $\boldsymbol{\xi}$ and $\mathbf{A}$ which satisfy (2·33) and (2·35), but not necessarily (2·32) and (2·34), and which make δW as given by (3·16) negative, then there is a $\tilde{\boldsymbol{\xi}}$ and and $\tilde{\mathbf{A}}$ satisfying (2·32) to (2·35) which make δW negative. Note that for the unrestricted $\boldsymbol{\xi}$ and $\mathbf{A}$, δW as given by (3·6) may differ from that given by (3·16) by the addition of terms which represent the work done at the surface against the unbalanced total pressure $\langle p+\frac{1}{2}|\mathbf{B}|^2\rangle$. Thus the form of δW given by (3·16) must be used for the extended principle.

In order to find $\tilde{\mathbf{A}}$ observe first that the Euler equation resulting from the minimization of δW_V ((3·17) with the constraint conditions (2·33) and (2·35)) is $\operatorname{curl}^2 \mathbf{A} = 0$ ((2·34)). Therefore, if $\mathbf{A}$ does not satisfy this equation, $\tilde{\mathbf{A}}$ can be chosen to satisfy it and certainly decrease δW_V thereby.

To complete the proof it remains to find $\tilde{\boldsymbol{\xi}}$. This is accomplished by modifying $\boldsymbol{\xi}$ by an infinitesimal amount. Let ϵ be a parameter of smallness and $\boldsymbol{\eta}$ a finite vector in the $\operatorname{grad} p$ direction which falls to zero in a distance ϵ as one moves normally away from the interface into the fluid. Write $\tilde{\boldsymbol{\xi}}$ as

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}+\epsilon\boldsymbol{\eta}. \tag{3·19}$$

To lowest order in ϵ

$$\operatorname{div}(\epsilon\boldsymbol{\eta}) = \mathbf{n}.[\mathbf{n}.\operatorname{grad}(\epsilon\boldsymbol{\eta})]-[\mathbf{n}\times(\mathbf{n}\times\operatorname{grad})].(\epsilon\boldsymbol{\eta}) \sim |\boldsymbol{\eta}|, \tag{3·20}$$

since $\boldsymbol{\eta}$ changes rapidly in the normal direction. Thus $\boldsymbol{\eta}$ can be chosen so that $\tilde{\boldsymbol{\xi}}$ satisfies (2·32). Furthermore

$$\begin{aligned}\delta W\{\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}\} &= \delta W\{\boldsymbol{\xi}+\epsilon\boldsymbol{\eta},\boldsymbol{\xi}+\epsilon\boldsymbol{\eta}\}\\ &= \delta W\{\boldsymbol{\xi},\boldsymbol{\xi}\}+O(\epsilon),\end{aligned} \tag{3·21}$$

since the integrands of $\delta W\{\boldsymbol{\xi},\epsilon\boldsymbol{\eta}\}$ and $\delta W\{\epsilon\boldsymbol{\eta},\epsilon\boldsymbol{\eta}\}$ are bounded and are different from zero only in a shell of thickness ϵ. Therefore, if $\delta W\{\boldsymbol{\xi},\boldsymbol{\xi}\}$ is negative ϵ can be chosen so small that $\delta W\{\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}\}$ is negative. It is clear that any $\boldsymbol{\xi}$ and $\mathbf{A}$ which do satisfy the conditions (2·32) and (2·34) can be considered to be members of the unrestricted class of $\boldsymbol{\xi}$ and $\mathbf{A}$. Thus, a necessary and sufficient condition for instability is that one can find a $\boldsymbol{\xi}$ and $\mathbf{A}$ which satisfy only (2·33) at a fluid-vacuum interface and (2·35) or (2·36) at a rigid, perfectly conducting boundary and make the potential energy, (3·16), negative. This completes the proof of the extended energy principle.

The above considerations are closely connected with Rayleigh's principle (Rayleigh 1877). In fact, it can be shown that the Euler equation of the variational principle

$$\omega^2 = \frac{\delta W\{\boldsymbol{\xi},\boldsymbol{\xi}\}}{K\{\boldsymbol{\xi},\boldsymbol{\xi}\}}, \quad \Delta\omega^2 = 0 \tag{3·22}$$

is just the eigenvalue equation (3·8). (Note that δ represents a variation due to a $\boldsymbol{\xi}$ deformation, while Δ is used to represent other variations.) If the form of δW is given by (3·6) then the variation in $\boldsymbol{\xi}$, $\Delta\boldsymbol{\xi}$, must satisfy (2·32). If, however, (3·16) is used for δW, then the variations $\Delta\mathbf{A}$ and $\Delta\boldsymbol{\xi}$ are subject only to equation (2·33), and (2·32) follows as a natural boundary condition.

The utility of Rayleigh's principle lies in the fact that when the ratio (3·22) possesses a minimum, it can be used to estimate oscillation frequencies or rates of growth of instability. For example, those displacements which make δW negative can be used as trial functions in the variational principle (3·22). Even when ω^2 is not bounded from below as is the case in certain hydromagnetic instabilities

(Kruskal & Schwarzschild 1954) Rayleigh's principle can still be employed to yield information on the structure and time constant of the eigenmodes.

In practice, the examination of the sign of δW in the energy principle is carried out in many cases by choosing a positive definite normalization condition on $\boldsymbol{\xi}$ and minimizing δW. This is formally similar to (3·22). The great advantage of the energy principle over both the normal mode technique and its equivalent, Rayleigh's principle, lies in the fact that one is not restricted to the normalization condition $K\{\boldsymbol{\xi}, \boldsymbol{\xi}\} = 1$, but can choose any convenient condition. Of course, in changing the normalization condition one loses knowledge of the exact eigenfrequencies but often gains the advantages of great analytical simplification. In § 6, there appear examples of alternative normalization conditions.

(a) *Extension of the energy principle to more general cases*

The description of a plasma given above may be inadequate if any of the conditions of validity (i) to (vii) of § 2 do not hold. In many cases of interest condition (iv), that the stress tensor be isotropic, and part of condition (vii), that the ion pressure gradient term and the gravitational potential term in Ohm's law be negligible, are not satisfied. In this section the formalism is generalized to include situations in which these conditions are no longer valid.

The new governing equations will be stated and δW derived. It will be found that the inclusion of the new terms in Ohm's law does not lead to a change in the formula for δW, when the stress tensor is isotropic.

In these more general cases, the equation of motion (2·1) is

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \operatorname{div} \overleftrightarrow{\mathbf{p}} - \rho \operatorname{grad} \phi \tag{3·23}$$

and the valid Ohm's law, replacing (2·3) is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{M}{e\rho} \operatorname{div} \overleftrightarrow{\mathbf{p}}_i - \frac{M}{e} \operatorname{grad} \phi = 0, \tag{3·24}$$

where $\overleftrightarrow{\mathbf{p}}$ is the total material stress tensor and $\overleftrightarrow{\mathbf{p}}_i$ is the ion partial stress tensor.

To derive an equation of state for the case of an anisotropic stress tensor, consider situations where the magnetic field is so strong that its change over an ion Larmor radius is small. Then the matter stress tensor $\overleftrightarrow{\mathbf{p}}$ is approximately diagonal in a local Cartesian co-ordinate system one of whose axes is directed along $\mathbf{B}$, and is invariant under rotations about $\mathbf{B}$. That is, if $\mathbf{e}$ denotes a unit vector parallel to $\mathbf{B}$ and $\overleftrightarrow{\mathbf{1}}$ the unit dyadic,

$$\overleftrightarrow{\mathbf{p}} = p_\perp(\overleftrightarrow{\mathbf{1}} - \mathbf{ee}) + p_\parallel \mathbf{ee}. \tag{3·25}$$

The internal energy per unit volume is given by one-half the trace of the stress tensor. Thus, the internal energy per unit mass can be written

$$u = u_\parallel + u_\perp, \tag{3·26}$$

where
$$u_\parallel = \frac{p_\parallel}{2\rho}, \tag{3·27}$$

and
$$u_\perp = \frac{p_\perp}{2\rho}. \tag{3·28}$$

If collisions are infrequent, $u_{\parallel}$ and $u_{\perp}$ are independent. Assume that there is no flow of heat and consider an element of mass contained in the element of volume $\mathrm{d}\tau = \mathrm{d}L\,\mathrm{d}S$, where $\mathrm{d}L$ is an element of length along **B** and $\mathrm{d}S$ an element of area perpendicular to **B**. In a displacement $\boldsymbol{\xi}$ the associated fractional change in length along a line of force is easily seen to be

$$\begin{aligned}\frac{\delta \mathrm{d}L}{\mathrm{d}L} &= \frac{1}{\mathrm{d}L}\mathbf{e}.[\boldsymbol{\xi}(\mathbf{r}+\mathbf{e}\,\mathrm{d}L)-\boldsymbol{\xi}(\mathbf{r})] \\ &= (\mathbf{e}.\operatorname{grad}\boldsymbol{\xi}).\mathbf{e}. \end{aligned} \tag{3·29}$$

The corresponding fractional change in area perpendicular to **B** is readily computed by observing that it follows from $\mathrm{d}\tau = \mathrm{d}L\,\mathrm{d}S$ and $\delta\,\mathrm{d}\tau/\mathrm{d}\tau = \operatorname{div}\boldsymbol{\xi}$ that

$$\frac{\delta \mathrm{d}L}{\mathrm{d}L}+\frac{\delta \mathrm{d}S}{\mathrm{d}S} = \frac{\delta \mathrm{d}\tau}{\mathrm{d}\tau} = \operatorname{div}\boldsymbol{\xi}; \tag{3·30}$$

whence

$$\frac{\delta \mathrm{d}S}{\mathrm{d}S} = \operatorname{div}\boldsymbol{\xi}-(\mathbf{e}.\operatorname{grad}\boldsymbol{\xi}).\mathbf{e}. \tag{3·31}$$

Thus if there is no heat flow in the course of the displacement, that is, if the displacement is locally adiabatic, one can write

$$\delta(u_{\parallel}\rho\,\mathrm{d}\tau) = \delta(\tfrac{1}{2}p_{\parallel}\,\mathrm{d}\tau) = -p_{\parallel}\,\mathrm{d}S\,\delta\,\mathrm{d}L, \tag{3·32}$$

$$\delta(u_{\perp}\rho\,\mathrm{d}\tau) = \delta(p_{\perp}\,\mathrm{d}\tau) = -p_{\perp}\,\mathrm{d}L\,\delta\,\mathrm{d}S. \tag{3·33}$$

The terms on the right above represent the external work done. From these expressions follow immediately the equations of state.

$$\left.\begin{aligned}\frac{\delta p_{\parallel}}{p_{\parallel}} &= -\operatorname{div}\boldsymbol{\xi}-2(\mathbf{e}.\operatorname{grad}\boldsymbol{\xi}).\mathbf{e}, \\ \frac{\delta p_{\perp}}{p_{\perp}} &= -2\operatorname{div}\boldsymbol{\xi}+(\mathbf{e}.\operatorname{grad}\boldsymbol{\xi}).\mathbf{e}.\end{aligned}\right\} \tag{3·34}$$

These equations agree with those found by Chew, Goldberger & Low (1956) by an analysis of the Boltzmann equation, employing somewhat different assumptions.

In order to derive the expression for the change in **B** due to a displacement $\boldsymbol{\xi}$, consider motions about a configuration of static equilibrium. For clarity the subscript zero is reintroduced to indicate equilibrium quantities. The equilibrium electric field is

$$\mathbf{E}_0 = \frac{M}{e}\operatorname{grad}_0\phi_0+\frac{M}{e\rho_0}\operatorname{div}_0\overleftrightarrow{\mathbf{p}}_{i,0}. \tag{3·35}$$

Since $\mathbf{E}_0$ is an electrostatic field its curl must vanish which implies that the right-hand side of (3·35) is the gradient of a scalar.

Assume that (3·34) holds with $p_{\parallel}$ and $p_{\perp}$ replaced by $p_{i\parallel}$ and $p_{i\perp}$, and note that in order of magnitude $p_{i\parallel}\sim p_{i\perp}\sim \rho kT_i/M$. Then the change in magnitude of

$$(M\operatorname{div}\overleftrightarrow{\mathbf{p}}_i)/e\rho$$

in a displacement $\boldsymbol{\xi}$ from equilibrium, which is not necessarily small, is approximately

$$\frac{M}{e}\frac{kT_i}{ML}\frac{\xi}{L}, \tag{3·36}$$

where L is a characteristic length over which the various physical quantities change. The corresponding change in the magnitude of $\mathbf{v}\times\mathbf{B}$ is

$$\omega\xi B, \tag{3·37}$$

where $1/\omega$ is a characteristic time of the motion. The ratio of formula (3·36) to formula (3·37) is

$$\frac{kT_i/M}{\omega\omega_{ci}L^2}, \tag{3·38}$$

where $\omega_{ci} = eB/M$ is the ion cyclotron frequency. For many systems of interest $\omega^2L^2 \sim kT_i/M$, while $\omega \ll \omega_{ci}$ by condition (vii) of § 2. Thus the ratio (3·38) is much less than unity and the ion stress tensor has negligible effect in determining the change in $\mathbf{E}$ from its equilibrium value, although it may play an important role in determining $\mathbf{E}_0$.

The change in $\mathbf{B}$, however, is determined from $\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t$. Thus it follows from the Ohm's law equation (3·24), neglecting the contribution of the term in $\operatorname{div}\overleftrightarrow{\mathbf{p}}_i$ on the basis of the preceding considerations, that

$$\frac{\partial\mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}). \tag{3·39}$$

Equation (3·39), however, is precisely what one obtains on combining the induction equation (2·5) with the Ohm's law of the preceding work, (2·3). Thus, in those cases where the stress tensor is isotropic, the linearized equations governing the motion are unchanged by the inclusion in the Ohm's law of the two additional terms. Therefore, $\mathbf{F}\{\boldsymbol{\xi}\}$ and δW are also unchanged and the energy principle holds in the form previously derived.

If the stress tensor is given by (3·25) and (3·34) there exists an energy integral

$$U = \int d\tau\{\tfrac{1}{2}\rho\,|\,\mathbf{v}\,|^2+\tfrac{1}{2}\,|\,\mathbf{B}\,|^2+p_\perp+\tfrac{1}{2}p_\parallel+\rho\phi\}, \tag{3·40}$$

while (3·34) (3·39), and the law of conservation of mass $\dot\rho = -\rho\operatorname{div}\mathbf{v}$ permit one to express $\overleftrightarrow{\mathbf{p}}$, $\mathbf{B}$ and ρ in terms of their initial values and $\boldsymbol{\xi}$. The expressions do not involve $\dot{\boldsymbol{\xi}}$. Thus, since the system is conservative there must exist a potential energy δW quadratic in $\boldsymbol{\xi}$ which implies as before that the associated first order force $\mathbf{F}\{\boldsymbol{\xi}\}$ is self-adjoint. The energy principle is, therefore, still valid and stability can be determined by examining the sign of the new δW which is given by

$$\begin{aligned}
\delta W &= -\frac{1}{2}\int d\tau\,\boldsymbol{\xi}\,.\,\mathbf{F}\{\boldsymbol{\xi}\}\\
&= \frac{1}{2}\int d\hat{\tau}\,|\operatorname{curl}\mathbf{A}\,|^2\\
&\quad -\frac{1}{2}\int d\sigma\{(\mathbf{n}\,.\,\boldsymbol{\xi})^2\,\mathbf{n}\,.\,\langle\operatorname{grad}(p_\perp+\tfrac{1}{2}\,|\,\mathbf{B}\,|^2)\rangle\\
&\qquad\qquad +\mathbf{e}\,.\,\boldsymbol{\xi}(p_\parallel-p_\perp)\,\mathbf{n}\,.\,[(\mathbf{e}\,.\operatorname{grad})\,\boldsymbol{\xi}-(\boldsymbol{\xi}\,.\operatorname{grad})\,\mathbf{e}]\}\\
&\quad +\frac{1}{2}\int d\tau\{|\,\mathbf{Q}\,|^2-\mathbf{j}\,.\,\mathbf{Q}\times\boldsymbol{\xi}+\tfrac{5}{3}p_\perp(\operatorname{div}\boldsymbol{\xi})^2+(\operatorname{div}\boldsymbol{\xi})\,(\boldsymbol{\xi}\,.\operatorname{grad})\,p_\perp\\
&\quad +\tfrac{1}{3}p_\perp[\operatorname{div}\boldsymbol{\xi}-3q]^2+q\operatorname{div}[\boldsymbol{\xi}(p_\parallel-p_\perp)]-[\operatorname{div}(\rho\boldsymbol{\xi})]\,\boldsymbol{\xi}\,.\operatorname{grad}\phi\\
&\quad -(p_\parallel-p_\perp)\,[\mathbf{e}\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,\mathbf{e}-\boldsymbol{\xi}\,.\,(\operatorname{grad}\mathbf{e})\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,\mathbf{e}\\
&\quad -4q^2+\mathbf{e}\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,(\mathbf{e}\,.\operatorname{grad}\boldsymbol{\xi})-\boldsymbol{\xi}\,.\,(\operatorname{grad}\mathbf{e})\,.\,(\mathbf{e}\,.\operatorname{grad}\boldsymbol{\xi})]\},
\end{aligned} \tag{3·41}$$

where $q = (\mathbf{e}\,.\operatorname{grad}\boldsymbol{\xi})\,.\,\mathbf{e}$ and the subscript zero distinguishing equilibrium quantities has been suppressed.

The boundary condition on $\mathbf{A}$ remains as before, (2·33). The jump condition on the pressure, (2·8), is replaced by

$$\langle p_\perp + \tfrac{1}{2} |\mathbf{B}|^2 \rangle = 0. \tag{3·42}$$

In some cases collisions are sufficiently frequent to yield an isotropic stress tensor for the equilibrium, but the collision time is much greater than an oscillation or instability time. Under such circumstances the stress tensor will not remain isotropic in the course of a motion but will be determined by (3·34), with $p_\parallel = p_\perp = p$. Expression (3·41) for δW then differs by a positive definite term from the corresponding equation (3·13) for the case where the stress tensor remains isotropic in the course of a motion with $\gamma = \frac{5}{3}$. Hence, the equilibrium is at least as stable.

(b) *Comparison theorems*

There are various comparison theorems which follow from the energy principle. Two examples will now be given.

Consider a system (I), a part of which is a vacuum region (a). Compare this with a system (II), which in the equilibrium state is identical with (I), except that the part corresponding to (a) is a zero-pressure plasma. Then if system (II) is unstable, so is system (I). To demonstrate this it is merely necessary to note that the expressions for δW for the two systems differ only in that the vacuum contribution $\frac{1}{2}\int d\tau\, |\operatorname{curl} \mathbf{A}|^2$ for region (a) of system (I) is replaced by $\frac{1}{2}\int |\operatorname{curl}(\boldsymbol{\xi} \times \mathbf{B})|^2 d\tau$ for system (II). Suppose $\boldsymbol{\xi}_{\mathrm{II}}$ and $\mathbf{A}_{\mathrm{II}}$ are trial functions which make the change in potential energy for system (II) negative. Then for system (I) choose $\mathbf{A}_{\mathrm{I}} = \mathbf{A}_{\mathrm{II}}$ and $\boldsymbol{\xi}_{\mathrm{I}} = \boldsymbol{\xi}_{\mathrm{II}}$ except in region (a) and there choose $\mathbf{A}_{\mathrm{I}} = \boldsymbol{\xi}_{\mathrm{II}} \times \mathbf{B}$, which is a valid trial function, since it satisfies the boundary conditions on $\mathbf{A}$. This choice makes δW for (I) also negative.

A second comparison theorem is established by considering two equilibria; case (I), a fluid region in contact with a surrounding vacuum region which in turn is enclosed by a rigid perfectly conducting wall; case (II), a fluid region which is identical with the fluid region of I, but is in contact with a surrounding vacuum region enclosed in a rigid perfectly conducting wall which either coincides with or is exterior to that of (I). Assume further that all equilibrium quantities are identical in the common regions of (I) and (II).

Suppose that vector fields $\boldsymbol{\xi}$ and $\mathbf{A}$ have been found which make δW negative for case (I). The vector potential $\mathbf{A}$ can be assumed to vanish identically on the rigid perfectly conducting wall enclosing (I) because of (2·35) and the fact that an arbitrary gradient can be added to $\mathbf{A}$ without changing δW. Clearly the same vector fields can be employed as trial functions for (II) with A chosen to be zero in any regions not common to both systems. Thus system (II) is certainly no more stable than (I).

4. Application of the energy principle

(a) *Procedure*

The energy principle shows that the question of stability of an equilibrium situation is reduced to an examination of the sign of $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ for arbitrary displacements $\boldsymbol{\xi}$. For some equilibria physical reasoning leads to $\boldsymbol{\xi}$'s which make

$\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ negative, and thus settles the question of stability in a simple manner. An example of this kind is given in § 5. In general, however, it is not possible immediately to exhibit such a $\boldsymbol{\xi}$. In this case a procedure is needed for examining $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ for all admissible $\boldsymbol{\xi}$'s in a systematic fashion. One tries to make $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ as small as possible. Since it is a homogeneous quadratic form in $\boldsymbol{\xi}$, one must introduce a condition to keep its values bounded from below. This condition can be chosen in any convenient way so long as it does not affect the sign of $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$. In particular it can be chosen to lead to analytical simplicity in the minimization. For example, one can impose normalization requirements like $\int d\tau_0 \rho_0 \boldsymbol{\xi}^2 = 1$, or alternatively one can prescribe $\mathbf{n}_0 . \boldsymbol{\xi}$ on the fluid vacuum boundary (where a subscript zero as usual denotes equilibrium quantities). In the latter case, it is, of course, necessary to minimize separately for all admissible prescriptions of $\mathbf{n}_0 . \boldsymbol{\xi}$.

Consider a plasma surrounded by a vacuum region. A convenient program for minimization consists of first examining $\boldsymbol{\xi}$'s which do not move the interface (i.e. $\mathbf{n}_0 . \boldsymbol{\xi} = 0$ on the interface). Note that with this boundary condition the surface terms do not contribute to δW and the non-negative vacuum term is minimized to zero by choosing $\mathbf{A} = 0$. If δW can be made negative, be it by inspection or by choosing a normalization condition and minimizing, then the equilibrium is unstable.

Suppose, however, δW is non-negative with the above boundary condition $\mathbf{n}_0 . \boldsymbol{\xi} = 0$. The equilibrium still may not be stable since displacements which move the boundary may yield a decrease in potential energy. In this case it is convenient to proceed by prescribing $\mathbf{n}_0 . \boldsymbol{\xi}$ (not everywhere zero) on the fluid-vacuum boundary, and minimizing δW_V and δW_F separately. No volume condition like $\int d\tau_0 \rho_0 \boldsymbol{\xi}^2 = 1$ is imposed here. Since δW_V is a non-negative form whose Euler equation is

$$\operatorname{curl}_0 \operatorname{curl}_0 \mathbf{A} = 0 \tag{4·1}$$

it obviously has a minimum.

Assume further, as is often true in practice, that there is a displacement $\boldsymbol{\xi}$ which makes δW_F stationary subject to a given prescription of $\mathbf{n}_0 . \boldsymbol{\xi}$. Then this stationary value must be an absolute minimum and thus unique. To show this let $\boldsymbol{\eta}$ be any displacement which satisfies the boundary condition $\mathbf{n}_0 . \boldsymbol{\eta} = 0$. Then

$$\delta W_F\{\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{\xi}+\boldsymbol{\eta}\} = \delta W_F\{\boldsymbol{\xi}, \boldsymbol{\xi}\} + 2\delta W_F\{\boldsymbol{\xi}, \boldsymbol{\eta}\} + \delta W_F\{\boldsymbol{\eta}, \boldsymbol{\eta}\}. \tag{4·2}$$

The assumption that $\boldsymbol{\xi}$ makes δW_F stationary requires that $\delta W_F\{\boldsymbol{\xi}, \boldsymbol{\eta}\} = 0$, and leads to the Euler equation

$$\mathbf{F}\{\boldsymbol{\xi}\} = 0. \tag{4·3}$$

Now, since $\mathbf{n}_0 . \boldsymbol{\eta} = 0$, $\delta W_F\{\boldsymbol{\eta}, \boldsymbol{\eta}\}$ is non-negative by supposition. Thus $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ is a minimum.

Form the scalar product of (4·1) with $\mathbf{A}$, and of (3·3) with $\boldsymbol{\xi}$, and integrate over their respective volumes. The resulting minimum potential energy, subject to the prescribed boundary values $\mathbf{n}_0 . \boldsymbol{\xi}$, is

$$\delta W = \frac{1}{2}\int d\sigma_0 \mathbf{n}_0 . \boldsymbol{\xi}\{\gamma p_0 \operatorname{div}_0 \boldsymbol{\xi} - \mathbf{B}_0 . \mathbf{Q}_0 - \mathbf{B}_0 . (\boldsymbol{\xi} . \operatorname{grad}_0 \mathbf{B}_0) + \hat{\mathbf{B}}_0 . \operatorname{curl} \mathbf{A} + \hat{\mathbf{B}}_0 . (\boldsymbol{\xi} . \operatorname{grad}_0 \hat{\mathbf{B}}_0)\}. \tag{4·4}$$

This expression, of course, represents the work done against the unbalanced first order total pressure $\langle p+\frac{1}{2}\,|\,\mathbf{B}\,|^2\rangle$ in a displacement of the boundary. Note that δW in (4·4) is a functional of $\mathbf{n}_0.\boldsymbol{\xi}$. The program is completed by minimizing (4·4) with respect to $\boldsymbol{\xi}.\mathbf{n}_0$.

(b) *A physical interpretation*

The problem of minimizing the volume contribution δW_F subject to the boundary condition $\mathbf{n}_0.\boldsymbol{\xi} = 0$, under a particular normalization, yields conditions of physical interest on the minimizing $\boldsymbol{\xi}$ when $\nabla\phi \equiv 0$. These conditions are that to first order in $\boldsymbol{\xi}$ the fields $\mathbf{j}$ and $\mathbf{B}$ are tangent to the surfaces p = constant. That this is true to zero order in $\boldsymbol{\xi}$, that is, for the equilibrium quantities, follows from (2·20).

The choice of normalization for the demonstration is motivated by the fact that it is possible by judicious integration by parts to write

$$\begin{aligned}\delta W_F = \frac{1}{2}\int \mathrm{d}\tau_0\{&|\,\mathbf{Q}_0+\mathbf{n}_0.\boldsymbol{\xi}\,\mathbf{j}_0\times\mathbf{n}_0\,|^2\\ &+\gamma p_0(\mathrm{div}_0\,\boldsymbol{\xi})^2\\ &-2(\mathbf{n}_0.\boldsymbol{\xi})^2\,\mathbf{j}_0\times\mathbf{n}_0.(\mathbf{B}_0.\mathrm{grad}_0\,\mathbf{n}_0)\},\end{aligned} \tag{4·5}$$

where $\mathbf{n}_0$ is the unit vector normal to the surface p_0 = constant. It is obvious from (4·5) that a normalizing condition involving $\mathbf{n}_0.\boldsymbol{\xi}$ alone (e.g. $\int \mathrm{d}\tau_0\,\rho_0(\mathbf{n}_0.\boldsymbol{\xi})^2 = 1$) should be sufficient to bound $\delta W_F\{\boldsymbol{\xi},\boldsymbol{\xi}\}$ from below. Let $\boldsymbol{\xi}$ minimize δW with such a normalizing condition. Any small change in $\boldsymbol{\xi}$, $\Delta\boldsymbol{\xi}$, must leave δW stationary, if it leaves the norm stationary. From the self-adjoint nature of $\mathbf{F}$, it follows that

$$\Delta[\delta W] = -\int \mathrm{d}\tau_0\,\Delta\boldsymbol{\xi}.\mathbf{F}\{\boldsymbol{\xi}\} = 0. \tag{4·6}$$

Consider $\Delta\boldsymbol{\xi}$'s of the form
$$\Delta\boldsymbol{\xi} = \Delta b\,\mathbf{j}_0+\Delta c\,\mathbf{B}_0, \tag{4·7}$$

where Δb and Δc are arbitrary since the normalization condition involves only $\boldsymbol{\xi}.\mathrm{grad}_0\,p_0$, and $\mathbf{j}_0$ and $\mathbf{B}_0$ are orthogonal to $\mathrm{grad}_0\,p_0$. Then it follows that the coefficients of Δb and Δc in the integrand of equation (4·6) must separately vanish. Now $\mathbf{F}\{\boldsymbol{\xi}\} = \{-\mathrm{grad}\,p+\mathbf{j}\times\mathbf{B}\}_1$, where the subscript unity means the part first order in $\boldsymbol{\xi}$, so

$$\mathbf{B}_0.\{-\mathrm{grad}\,p+\mathbf{j}\times\mathbf{B}\}_1 = 0, \tag{4·8}$$

$$\mathbf{j}_0.\{-\mathrm{grad}\,p+\mathbf{j}\times\mathbf{B}\}_1 = 0. \tag{4·9}$$

Note, however, that it follows on taking the first order part of the identities $\mathbf{B}.\mathbf{j}\times\mathbf{B} = 0$ and $\mathbf{j}.\mathbf{j}\times\mathbf{B} = 0$, and using $\mathbf{j}_0\times\mathbf{B}_0 = (\mathrm{grad}\,p)_0$, that

$$\mathbf{B}_0.(\mathbf{j}\times\mathbf{B})_1+\mathbf{B}_1.(\mathrm{grad}\,p)_0 = 0, \tag{4·10}$$

$$\mathbf{j}_0.(\mathbf{j}\times\mathbf{B})_1+\mathbf{j}_1.(\mathrm{grad}\,p)_0 = 0. \tag{4·11}$$

Thus if one subtracts (4·10) and (4·11), respectively, from (4·8) and (4·9), there results, correct to first order in $\boldsymbol{\xi}$,

$$(\mathbf{B}_0+\mathbf{B}_1).[(\mathrm{grad}\,p)_0+(\mathrm{grad}\,p)_1] = 0, \tag{4·12}$$

$$(\mathbf{j}_0+\mathbf{j}_1).[(\mathrm{grad}\,p)_0+(\mathrm{grad}\,p)_1] = 0. \tag{4·13}$$

Equations (4·12) and (4·13) express the conditions stated earlier, that to first order in $\boldsymbol{\xi}$ the fields $\mathbf{j}$ and $\mathbf{B}$ are tangent to the surfaces p = constant.

After some manipulation, (4·12) (or equivalently (4·8)) can be rewritten in the form

$$\mathbf{B}_0 . \operatorname{grad}_0 \operatorname{div}_0 \boldsymbol{\xi} = 0, \tag{4·14}$$

which is often useful in practice.

5. Stability of a plasma with no internal magnetic field

Consider a plasma in which the magnetic field vanishes and the pressure is constant and outside which there is a vacuum region with a magnetic field. Let $\phi = 0$. It was suggested by E. Teller (1954, private communication) on intuitive grounds that if the lines of force on the interface are anywhere concave to the plasma the state is unstable to local displacements. This is readily demonstrated using the energy principle.

Choose a divergence-free displacement $\boldsymbol{\xi}$ so that

$$2\delta W = \int d\hat{\tau} \,|\operatorname{curl} \mathbf{A}|^2 - \frac{1}{2}\int d\sigma \,(\hat{\mathbf{n}}.\boldsymbol{\xi})^2 \,\hat{\mathbf{n}}.\operatorname{grad} |\hat{\mathbf{B}}|^2, \tag{5·1}$$

where $\hat{\mathbf{n}}$ is the normal to the interface pointing towards the plasma. Denote by $\mathbf{R}$ the vector from a point on a line of force to the centre of curvature of the line. Since, with $|\mathbf{R}| = R$,

$$\tfrac{1}{2}\hat{\mathbf{n}}.\operatorname{grad} |\mathbf{B}|^2 = \hat{\mathbf{n}}.\mathbf{R}\frac{|\mathbf{B}|^2}{R^2}, \tag{5·2}$$

the surface term in (5·1) is negative or positive according to whether or not $\mathbf{R}$ points towards the plasma. If $\mathbf{R}$ everywhere points away from the plasma, δW is obviously positive for all $\boldsymbol{\xi}$ and $\mathbf{A}$ (even if $\operatorname{div}\boldsymbol{\xi} \neq 0$) and the system is stable.

Consider a point on the interface where $\mathbf{R}$ is directed towards the plasma and construct a local Cartesian co-ordinate system in a small region about this point, with the z axis normal to the surface and pointing into the vacuum, and the x axis in the direction of $\mathbf{B}$. Choose the trial displacement $\boldsymbol{\xi}$ so that

$$\xi_z(x, y, 0) = \xi_0 f(x, y) \sin ky, \tag{5·3}$$

where f is a function of order unity which falls to zero in the small distance $a \ll R$ and where $ka^2 \gg R$. Choose also the trial vector potential

$$\mathbf{A}(x, y, z) = f(x, y) \operatorname{grad}\left(\frac{\xi_0 B}{k} \cos ky \,e^{-kz}\right), \tag{5·4}$$

which satisfies boundary condition (2·33) where

$$\mathbf{B} = B\mathbf{e}_x. \tag{5·5}$$

These choices make the vacuum contribution to δW negligible compared to the surface term. For

$$\int d\hat{\tau}\,|\operatorname{curl}\mathbf{A}|^2 = \int d\hat{\tau}\left\{\operatorname{grad} f \times \operatorname{grad}\left[\frac{\xi_0 B}{k}\cos ky\,e^{-kz}\right]\right\}^2$$

$$\approx \int d\hat{\tau}\,|\operatorname{grad} f|^2 \xi_0^2 B^2 e^{-2kz} \approx \frac{\xi_0^2 B^2}{2k}, \tag{5·6}$$

while

$$\int d\sigma\,(\hat{\mathbf{n}}.\boldsymbol{\xi})^2\,\hat{\mathbf{n}}.\mathbf{R}\frac{|\hat{\mathbf{B}}|^2}{R^2} \approx \frac{1}{R}\xi_0^2 B^2 a^2. \tag{5·7}$$

Therefore, δW is negative and by the energy principle the system is unstable.

Note that the deformation which produces instability tends to flute the surface along the lines of force. This moves some of the magnetic lines of force into a region previously occupied by matter and thus shortens them while only slightly bending them. The result is a decrease in the magnetic energy with no change in the gas energy.

Similar results have been obtained independently by H. Grad (1955) and C. Longmire (1955) (both private communications).

To estimate the rate of growth of this instability in the plasma choose the displacement

$$\xi_x = 0, \quad \xi_y = \xi_0 \surd f \cos ky\, \mathrm{e}^{kz}, \quad \xi_z = \xi_0 \surd f \sin ky\, \mathrm{e}^{kz}. \tag{5·8}$$

This $\boldsymbol{\xi}$ satisfies $\operatorname{div} \boldsymbol{\xi} = 0$ to order $(ka)^{-1}$. Then the kinetic energy form is

$$2K = \int \mathrm{d}\tau \rho \,|\, \boldsymbol{\xi} \,|^2 \approx \frac{\rho \xi_0^2 a^2}{2k}$$

and

$$\omega^2 = \frac{\delta W}{K} \approx -\frac{2B^2}{\rho} \frac{k}{R}. \tag{5·9}$$

This is unbounded as k approaches infinity.

Gravitational effects are readily included in this case if we assume the fluid to have constant density but varying pressure in the equilibrium state. This situation is an extension of the hydromagnetic Rayleigh–Taylor problem (Kruskal & Schwarzschild 1954) to an arbitrary interface. Choose the trial functions $\boldsymbol{\xi}$ and $\mathbf{A}$ as before. The surface term in (5·1) is modified by the addition of the term

$$\int \mathrm{d}\sigma \, (\hat{\mathbf{n}} . \boldsymbol{\xi})^2 \, \hat{\mathbf{n}} . \operatorname{grad} p \tag{5·10}$$

which, in virtue of the equilibrium relation

$$\operatorname{grad} p = -\rho \operatorname{grad} \phi, \tag{5·11}$$

becomes

$$-\int \mathrm{d}\sigma \, (\hat{\mathbf{n}} . \boldsymbol{\xi})^2 \rho \hat{\mathbf{n}} . \operatorname{grad} \phi. \tag{5·12}$$

The calculation now goes through as before and the situation is unstable if

$$\hat{\mathbf{n}} . \left[\mathbf{R} \frac{|\, \hat{\mathbf{B}} \,|^2}{R^2} + \rho \operatorname{grad} \phi \right] > 0 \tag{5·13}$$

anywhere on the boundary. In the case of a plane interface $\mathbf{R}$ is infinite and the familiar hydromagnetic Rayleigh–Taylor instability criterion is recovered.

6. Stability of an axisymmetric system

A more general case than that treated in the previous section occurs when a magnetic field may be present in the plasma. This situation can be treated exactly for two simple types of axisymmetric equilibrium situations. It is assumed that gravitational effects are negligible ($\phi = 0$).

The first consists of a longitudinal current giving rise to a toroidal magnetic field whose pressure supports a radial material pressure gradient. This is the well-known pinch effect (see, for example, Kruskal & Schwarzschild 1954).

The second consists of longitudinal and radial magnetic fields produced by currents in the azimuthal direction. Again, a radial material pressure gradient is supported by the magnetic field. The plasma is assumed to be in contact with a rigid perfectly conducting wall. This equilibrium is studied here. It is shown that it is possible to reduce the problem of stability to the consideration of an ordinary second-order differential equation of the Sturm–Liouville type. In fact, virtually all that is necessary is to find the number of negative eigenvalues which this equation possesses. In certain limiting cases one can further express the criterion for stability in terms of simple properties of the equilibrium.

Note that in the previous problem of § 5 either δW is obviously positive definite or one can easily display trial functions $\boldsymbol{\xi}$ and $\mathbf{A}$ which make it negative. In the problem of this section, however, it is necessary to examine the sign of δW for all possible displacements $\boldsymbol{\xi}$. This is accomplished by first writing δW in a co-ordinate system natural to the problem and then successively minimizing with respect to the components of the vector $\boldsymbol{\xi}$.

The equilibrium vector potential $\mathbf{A}_0$ in a fluid of this type (which is to be distinguished from the first order vacuum vector potential $\mathbf{A}$ previously introduced) has only an azimuthal component, since the current density j is itself azimuthal. Therefore, if in cylindrical co-ordinates (r, θ, z) one writes $\psi = rA_0(r, z)$, then

$$\mathbf{B} = \operatorname{curl}(\mathbf{e}_\theta \psi/r) = -(1/r)\,\mathbf{e}_\theta \times \operatorname{grad}\psi. \tag{6·1}$$

It follows from equation (6·1) that $\mathbf{B}.\operatorname{grad}\psi = 0$. Thus the lines of force lie in the surfaces $\psi =$ constant and also in the planes $\theta =$ constant. Moreover, if one chooses $\psi(0, z) = 0$, it is readily demonstrated that the magnetic flux interior to the surface $\psi =$ constant is $2\pi\psi$.

Because of this flux property, it is convenient to employ ψ as a co-ordinate. In order to retain an orthogonal co-ordinate system, introduce a function χ whose level surfaces are perpendicular to the surfaces $\psi =$ constant and $\theta =$ constant. Choose χ so that the set (ψ, θ, χ) forms a right-handed orthogonal system. Note that the volume element in this co-ordinate system is

$$d\tau = J\,d\psi\,d\theta\,d\chi, \tag{6·2}$$

where

$$1/J = B\,|\operatorname{grad}\chi| = \operatorname{grad}\psi.\operatorname{grad}\theta \times \operatorname{grad}\chi. \tag{6·3}$$

Thus

$$\operatorname{grad} = rB\mathbf{e}_\psi \frac{\partial}{\partial\psi} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \frac{1}{JB}\mathbf{e}_\chi \frac{\partial}{\partial\chi}, \tag{6·4}$$

where

$$\mathbf{e}_\psi = \frac{\operatorname{grad}\psi}{|\operatorname{grad}\psi|}, \tag{6·5}$$

$$\mathbf{e}_\theta = \frac{\operatorname{grad}\theta}{|\operatorname{grad}\theta|}, \tag{6·6}$$

$$\mathbf{e}_\chi = \frac{\operatorname{grad}\chi}{|\operatorname{grad}\chi|}. \tag{6·7}$$

Then, by (2·21) and (2·20),

$$\mathbf{j} = -\mathbf{e}_\theta \frac{r}{J}\frac{\partial}{\partial\psi}(JB^2) = j\mathbf{e}_\theta \tag{6·8}$$

and

$$\operatorname{grad} p = \mathbf{j} \times \mathbf{B} = -\mathbf{e}_\psi \frac{rB}{J}\frac{\partial}{\partial\psi}(JB^2). \tag{6·9}$$

Thus the pressure p is a function of ψ alone and if differentiation with respect to ψ is denoted by a prime, (6·9) can be written

$$(\ln JB^2)' = -p'/B^2. \tag{6·10}$$

Therefore

$$J = \frac{1}{B^2}\exp\left\{-\int_0^{\psi} \mathrm{d}\psi \frac{p'}{B^2}\right\}, \tag{6·11}$$

where the constant of integration (which is an arbitrary function of χ) has incidentally been chosen to make χ reduce to the magnetic scalar potential when $p = 0$.

Note that it follows from (6·8) and (6·9) that

$$p' = j/r \tag{6·12}$$

and j/r is constant along a line of force.

Using these results the potential energy for the system is

$$\begin{aligned} \delta W &= \delta W_F \\ &= \frac{1}{2}\int \mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\chi\, J\left\{\left[\frac{1}{rBJ}\frac{\partial}{\partial\chi}(rB\xi_\psi)\right]^2 + \left[\frac{r}{J}\frac{\partial}{\partial\chi}\left(\frac{\xi_\theta}{r}\right)\right]^2\right. \\ &\quad + B^2\left[\frac{\partial}{\partial\psi}(rB\xi_\psi) + \frac{\partial}{\partial\theta}\left(\frac{\xi_\theta}{r}\right)\right]^2 \\ &\quad + p'rB\xi_\psi\left[\frac{\partial}{\partial\psi}(rB\xi_\psi) + \frac{\partial}{\partial\theta}\left(\frac{\xi_\theta}{r}\right)\right] \\ &\quad + \frac{\gamma p}{J^2}\left[\frac{\partial}{\partial\psi}(rB\xi_\psi J) + \frac{\partial}{\partial\theta}\left(\frac{J\xi_\theta}{r}\right) + \frac{\partial}{\partial\chi}\left(\frac{\xi_\chi}{B}\right)\right]^2 \\ &\quad + \frac{p'rB\xi_\psi}{J}\left[\frac{\partial}{\partial\psi}(rB\xi_\psi J) + \frac{\partial}{\partial\theta}\left(\frac{J\xi_\theta}{r}\right)\right] \\ &\quad \left. + \frac{1}{J}\frac{\partial}{\partial\chi}(p'\xi_\chi\xi_\psi r)\right\}. \end{aligned} \tag{6·13}$$

Assume that the equilibrium quantities appearing in (6·13) are periodic over some fundamental period in χ which is equivalent to periodicity in z and also impose the boundary condition that $\boldsymbol{\xi}$ be periodic in χ over this period. All definite integrals with respect to χ are to be understood as extended over this period. The last term in (6·13) then integrates to zero.

Now proceed to minimize δW over all displacements $\boldsymbol{\xi}$. First note that the integrand in (6·13) depends on θ only via $\boldsymbol{\xi}$. This suggests Fourier analysis of $\boldsymbol{\xi}$ with respect to θ. Write $\boldsymbol{\xi}$ in the form

$$\begin{aligned} \xi_\psi &= \sum_{m=0}^{\infty} \frac{1}{rB} X_m(\psi,\chi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta, \\ \xi_\theta &= \sum_{m=1}^{\infty} \frac{r}{m} Y_m(\psi,\chi)\begin{Bmatrix}\sin\\ -\cos\end{Bmatrix} m\theta + \xi_\theta^{(0)}(\psi,\chi), \\ \xi_\chi &= \sum_{m=0}^{\infty} BZ_m(\psi,\chi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\theta. \end{aligned} \tag{6·14}$$

The potential energy δW is to be minimized over the set $(X_m, Y_m, Z_m, \xi_\theta^{(0)})$.

Upon integration with respect to θ, the cross terms of the double series vanish and

$$\delta W = \delta W_0 + 2 \sum_{m=1}^{\infty} \delta W_m, \tag{6·15}$$

where

$$\begin{aligned}\delta W_m = \tfrac{1}{2}\pi \int \mathrm{d}\chi\,\mathrm{d}\psi \Bigg\{ &\frac{1}{r^2B^2J}\left(\frac{\partial X_m}{\partial \chi}\right)^2 + \frac{1}{m^2}\frac{r^2}{J}\left(\frac{\partial Y_m}{\partial \chi}\right)^2 \\ &+ B^2J\left(\frac{\partial X_m}{\partial \psi} + Y_m\right)^2 + p'JX_m\left(\frac{\partial X_m}{\partial \psi} + Y_m\right) \\ &+ \frac{\gamma p}{J}\left[\frac{\partial}{\partial \psi}(JX_m) + JY_m + \frac{\partial Z_m}{\partial \chi}\right]^2 \\ &+ p'X_m\left(J\frac{\partial X_m}{\partial \psi} + X_m\frac{\partial J}{\partial \psi} + JY_m\right)\Bigg\},\end{aligned} \tag{6·16}$$

and $\frac{1}{2}\delta W_0$ is obtained by replacing Y_m in (6·16) by $m\xi_\theta^{(0)}/r$ and setting $m = 0$.

Since for each m, δW_m depends only on the set (X_m, Y_m, Z_m), it can be varied independently. It is clear from (6·16) that if δW_m can be made negative then δW_{m+1} can also be made negative. Thus it suffices to consider only the limiting case $m = \infty$. Do so and suppress the subscript ∞. After some algebraic manipulation, (6·16) becomes

$$\begin{aligned}\delta W = \tfrac{1}{2}\pi \int \mathrm{d}\psi\,\mathrm{d}\chi\, J \Bigg\{ &\frac{1}{r^2B^2J^2}\left(\frac{\partial X}{\partial \chi}\right)^2 + p'X^2\frac{\partial \ln J}{\partial \psi} - \frac{p'^2X^2}{B^2} \\ &+ \frac{1}{B^{-2}+(\gamma p)^{-1}}\left(X\frac{\partial \ln J}{\partial \psi} + \frac{1}{J}\frac{\partial Z}{\partial \chi} - \frac{p'X}{B^2}\right)^2 \\ &+ (B^2+\gamma p)\left[Y + \frac{\partial X}{\partial \psi} + \frac{X(p'+\gamma p\,\partial \ln J/\partial \psi) + (\gamma p/J)\,\partial Z/\partial \chi}{B^2+\gamma p}\right]^2\Bigg\}.\end{aligned} \tag{6·17}$$

For arbitrary fixed trial functions X and Z the expression above is minimized with respect to Y by choosing

$$-Y = \frac{\partial X}{\partial \psi} + \left[X\left(p' + \gamma p\frac{\partial \ln J}{\partial \psi}\right) + \frac{\gamma p}{J}\frac{\partial Z}{\partial \chi}\right](B^2+\gamma p)^{-1}, \tag{6·18}$$

which makes

$$\delta W = \tfrac{1}{2}\pi \int \mathrm{d}\psi\,\mathrm{d}\chi \left\{\frac{1}{r^2B^2J}\left(\frac{\partial X}{\partial \chi}\right)^2 + p'DX^2J + \frac{J}{B^{-2}+(\gamma p)^{-1}}\left[XD + \frac{1}{J}\frac{\partial Z}{\partial \chi}\right]^2\right\}, \tag{6·19}$$

where

$$\begin{aligned} D = \frac{\partial \ln J}{\partial \psi} - \frac{p'}{B^2} &= -\frac{2}{B^2}\frac{\partial}{\partial \psi}(p + \tfrac{1}{2}B^2) \\ &= -\frac{2}{rB^3}\mathbf{e}_\psi.[\mathbf{B}.\mathrm{grad}\,\mathbf{B}]\end{aligned} \tag{6·20}$$

is positive or negative at a point χ, ψ according to whether the line of force through that point is concave or convex towards the side of smaller ψ. Consequently, the system can be unstable only if somewhere a line of force is concave toward the side of larger p.

Equation (6·18) corresponds to (4·13) of the general minimization scheme, the content of which is that the minimizing displacement is such that the perturbed current density $\mathbf{j}$ lies in the perturbed constant pressure surfaces.

Next, the Euler equation resulting from minimizing (6·19) with respect to Z for fixed X reads

$$\frac{\partial}{\partial \chi}\left[\frac{1}{B^{-2}+(\gamma p)^{-1}}\left(XD+\frac{1}{J}\frac{\partial Z}{\partial \chi}\right)\right] = 0. \tag{6·21}$$

This equation corresponds to (4·12), the content of which is that the perturbed lines of force lie in the perturbed constant pressure surfaces. Equation (6·21) yields on integration with respect to χ

$$\frac{\partial Z}{\partial \chi} = \left(\frac{1}{B^2}+\frac{1}{\gamma p}\right)Jf(\psi)-JDX. \tag{6·22}$$

The constant of integration $f(\psi)$ is determined by integrating (6·22) with respect to χ, namely,

$$f(\psi) = \frac{2\pi}{L'+V'/\gamma p}\int \mathrm{d}\chi JDX, \tag{6·23}$$

where $$L' = 2\pi\int \mathrm{d}\chi\frac{J}{B^2}, \quad V' = 2\pi\int \mathrm{d}\chi J.$$

Note that $V'\,\mathrm{d}\psi$ is the volume contained between two neighbouring constant ψ surfaces.

The minimum δW now is

$$\delta W = \tfrac{1}{2}\pi\int \mathrm{d}\psi\,\mathrm{d}\chi\left\{\frac{1}{r^2B^2J}\left(\frac{\partial X}{\partial \chi}\right)^2+p'JDX^2\right\}+\frac{1}{4}\int \mathrm{d}\psi f^2\left(L'+\frac{V'}{\gamma p}\right). \tag{6·24}$$

The integrands above do not contain any derivatives of X with respect to ψ. Thus one can consider ψ to be merely a parameter and write

$$\delta W = \int \mathrm{d}\psi\,\delta W(\psi), \tag{6·25}$$

where $\delta W(\psi)$ depends only on the values of X on the surface ψ. Consequently δW can be made negative if and only if $\delta W(\psi)$ can be made negative for some value of ψ.

As in § 4, it is necessary to normalize X to achieve a well-posed minimum problem. An analytically simple normalizing condition is

$$H = \tfrac{1}{2}\pi\int \mathrm{d}\chi JX^2 = 1.$$

The minimization of $\delta W(\psi)$ under this normalization is equivalent to minimizing

$$\Lambda = \frac{\delta W(\psi)}{H} = \frac{\dfrac{1}{2\pi}\left(L'+\dfrac{V'}{\gamma p}\right)f^2+\displaystyle\int \mathrm{d}\chi\left[\frac{1}{r^2B^2J}\left(\frac{\partial X}{\partial \chi}\right)^2+p'JDX^2\right]}{\displaystyle\int \mathrm{d}\chi JX^2}. \tag{6·26}$$

Note that L', V' and J are all positive and only the term involving D can make Λ negative. It is possible to derive a sufficient condition for instability from (6·26) by choosing X to be constant in χ. Then

$$\Lambda = \frac{\gamma p(V''-p'L')(V''/V'+p'/\gamma p)}{V'+\gamma pL'}, \tag{6·27}$$

and if for any value of ψ this expression is negative the system is unstable.

In certain limiting cases it is possible to derive necessary and sufficient stability criteria directly from (6·26). In general, however, one must proceed with the formal minimization program.

The Euler equation resulting from the minimization of Λ is

$$\frac{\partial}{\partial\chi}\left(\frac{1}{r^2B^2J}\frac{\partial X}{\partial\chi}\right)+(\Lambda-p'D)\,JX = JDf, \tag{6·28}$$

where the variation in f has been computed from (6·23).

It is possible to derive from (6·28) certain general criteria for stability by expanding its solutions in terms of the eigenfunctions X_j of the Sturm–Liouville equation

$$\frac{\partial}{\partial\chi}\left(\frac{1}{r^2B^2J}\frac{\partial X_j}{\partial\chi}\right)+(\lambda_j-p'D)\,X_jJ = 0, \tag{6·29}$$

obtained by omitting the integral on the right-hand side of (6·28).

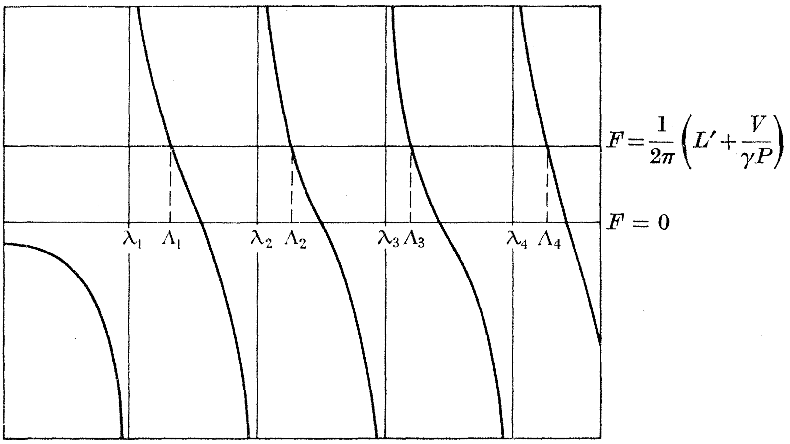

FIGURE 1. Schematic plot of $F(\Lambda)$ against Λ.

By the Sturmian theory (Ince 1944) the X_j comprise a complete set of eigenfunctions with associated eigenvalues λ_j. The λ_j are all distinct and can be arranged in an infinite increasing sequence $\lambda_1, \lambda_2, \ldots$. Note that the X_j can be normalized such that

$$\int d\chi\, X_i X_j J = \delta_{ij}. \tag{6·30}$$

Thus one can write

$$X = \Sigma b_j X_j, \tag{6·31}$$

$$D = \Sigma a_j X_j. \tag{6·32}$$

Then there results upon substitution in (6·28)

$$-\Sigma b_j(\lambda_j-\Lambda)\,JX_j = fJ\Sigma a_j X_j, \tag{6·33}$$

and in virtue of (6·30) it follows that

$$-b_j(\lambda_j-\Lambda) = a_j f. \tag{6·34}$$

But if one substitutes (6·31) and (6·32) into (6·23) and then employs (6·30) and (6·34) one finds

$$\frac{1}{2\pi}\left(L'+\frac{V'}{\gamma p}\right) = \Sigma\frac{a_j^2}{\Lambda-\lambda_j}. \tag{6·35}$$

The roots of (6·35) determine the possible values of Λ. Denote the right-hand side by $F(\Lambda)$ and plot it versus Λ. Note that $\mathrm{d}F/\mathrm{d}\Lambda < 0$. If none of the a_j is zero the graph is as in figure 1 and the intersections of this curve with the horizontal line $F(\Lambda) = (\frac{1}{2}\pi)(L' + V'/\gamma p)$ are the eigenvalues Λ_j of (6·28). If $a_j = 0$ for some j, the associated branch of $F(\Lambda)$ is not present in the diagram. It follows in this case from (6·34) that the associated root is $\Lambda = \lambda_j$. This is also the result which one would obtain if one considered the limit as $a_j \to 0$ of the associated intersection of the graph.

Clearly from figure 1, $\lambda_1 \leqslant \Lambda_1 \leqslant \lambda_2 \leqslant \Lambda_2 \ldots$. Thus if λ_1 is positive, so are all the Λ_j's while if λ_2 is negative, then Λ_1 is negative. If λ_1 is negative and λ_2 positive, then the sign of Λ_1 is not obvious. However, it is possible in this case to derive a criterion for the sign of Λ_1. Integrate (6·29) with respect to χ. There results

$$\lambda_j \int \mathrm{d}\chi J X_j = p' \int \mathrm{d}\chi J X_j \Sigma a_i X_i = p' a_j. \tag{6·36}$$

Thus

$$F(0) = -\Sigma \frac{a_i^2}{\lambda_i} = -\Sigma \frac{a_i}{p'} \int \mathrm{d}\chi J X_i = -\frac{1}{p'} \int \mathrm{d}\chi J D$$

$$= -\frac{1}{p'} \int \mathrm{d}\chi \left(J' - p' \frac{J}{B^2} \right) = -\frac{V''}{2\pi p'} + \frac{L'}{2\pi}. \tag{6·37}$$

Now assume that $\lambda_1 < 0$, $\lambda_2 > 0$. Since Λ_1 is determined by the condition

$$F(\Lambda_1) = \frac{1}{2\pi}(L' + V'/\gamma p)$$

and $F(\Lambda)$ is monotonically decreasing in the interval $\lambda_1 < \Lambda < \lambda_2 \ldots$, it is clear that if $F(0) > (1/2\pi)(L' + V'/\gamma p)$ then $\Lambda_1 > 0$ and conversely. But

$$F(0) - \frac{1}{2\pi}\left(L' + \frac{V'}{\gamma p} \right) = -\frac{V'}{2\pi p'} \left(\frac{V''}{V'} + \frac{p'}{\gamma p} \right). \tag{6·38}$$

One can write

$$\lambda_1 \geqslant 0 \to \Lambda_1 \geqslant 0,$$

$$\lambda_1 < 0 < \lambda_2 \to \Lambda_1 \gtreqless 0 \quad \text{as} \quad \frac{V'}{2\pi p'} \left(\frac{V''}{V'} + \frac{p'}{\gamma p} \right) \lesseqgtr 0,$$

$$\lambda_2 < 0 \to \Lambda_1 < 0. \tag{6·39}$$

In three limiting cases stability criteria can be obtained directly from (6·26), (i) if the material pressure is small compared with the magnetic pressure (i.e. $2p \ll B^2$), (ii) if the surface ψ = constant under consideration lies close to a cylinder and (iii) if the pressure gradient is large.

(a) *Case I*

Consider all quantities to be expanded in some parameter of smallness which essentially measures $2p/B^2$ and write

$$\left. \begin{aligned} p &= 0 + p^{(1)} + \ldots, & X &= X^{(0)} + X^{(1)} + \ldots, \\ B &= B^{(0)} + B^{(1)} + \ldots, & \Lambda &= \Lambda^{(0)} + \Lambda^{(1)} + \ldots, \end{aligned} \right\} \tag{6·40}$$

with similar expressions for other quantities. There results from (6·26) to lowest order,

$$\Lambda^{(0)} \int \mathrm{d}\chi J^{(0)} X^{(0)2} = \int \mathrm{d}\chi \frac{1}{[r^{(0)} B^{(0)}]^2 J^{(0)}} \left[\frac{\partial X^{(0)}}{\partial \chi} \right]^2. \tag{6·41}$$

Clearly $\Lambda^{(0)}$ is minimized by choosing $X^{(0)}$ constant in χ, which yields $\Lambda^{(0)} = 0$. This expresses the fact that the lowest order equilibrium is neither stable nor unstable, but neutral. Proceeding to the next order we find

$$\Lambda^{(1)}\int d\chi J^{(0)}X^{(0)2} = \frac{V^{(0)\prime}}{2\pi\gamma p^{(1)}}f^{(1)2}+\int \mathrm{d}\chi p^{(1)\prime}D^{(0)}J^{(0)}X^{(0)2}, \tag{6·42}$$

which by employing (6·23) can be reduced to

$$\Lambda^{(1)} = \gamma p^{(1)}\frac{V^{(0)\prime\prime}}{V^{(0)\prime}}\left[\frac{V^{(0)\prime\prime}}{V^{(0)\prime}}+\frac{p^{(1)\prime}}{\gamma p^{(1)}}\right]. \tag{6·43}$$

The sign of $\Lambda^{(1)}$ determines stability in this case. Equation (6·43) agrees with the criterion of (6·39) in the case $2p/B^2 \ll 1$, since if $V''p' > 0$, $\lambda_1 > 0$ and both equations yield stability, while if $V''p' < 0$, $\lambda_1 < 0 < \lambda_2$ and (6·43) agrees with the second part of (6·39).

(*b*) *Case II*

Consider a surface ψ = constant. Denote by R the radius of curvature of a line of force, by L the characteristic length for the variation of equilibrium quantities along a line of force, and by a the characteristic distance in which the pressure changes by an amount comparable with itself. Assume that everywhere on this surface ψ = constant,

$$L^2r/Ra^2 \ll 1, \tag{6·44}$$

in which circumstance the positive term in Λ proportional to $(\partial X/\partial\chi)^2$ dominates, unless $\partial X/\partial \chi = 0$ to lowest order in the parameter of smallness. Thus one is led to choose $X^{(0)}$ = constant. This leads immediately as in (6·27) to the first-order result

$$\Lambda = \gamma p(V''-p'L')(V''/V'+p'/\gamma p)(V''+\gamma pL')^{-1}. \tag{6·45}$$

Equation (6·45) reduces to (6·43) in the limit of small p. If $L^2r/Ra^2 \ll 1$ for all surfaces ψ = constant, then that (6·45) be negative on some surface is a necessary and sufficient condition for instability, otherwise it is only sufficient. Relation (6·44) is obviously satisfied if the surfaces are very nearly cylindrical.

Equation (6·20) gives an estimate as to the order of magnitude of R. If the two terms in the first line of (6·20) do not cancel, one obtains

$$R \sim a^2/r. \tag{6·46}$$

However, if they do cancel, as in the case of the cylinder, R is an order of magnitude larger. With this reservation, (6·44) reduces to

$$r \ll a^2/L. \tag{6·47}$$

Equation (6·45) is thus valid, for any equilibrium, for ψ surfaces close enough to the cylindrical axis.

(*c*) *Case III*

Consider an equilibrium such that everywhere on some surface ψ = constant

$$|\operatorname{grad} p| \gg B^2R/S^2, \tag{6·48}$$

where R again is the magnitude of the radius of curvature of a line of force and S is the distance over which it has the same sign. Assume that there is some region on this surface for which $p'D < 0$ and construct a trial function X which is zero outside of this region and varies smoothly within it. Then inequality (6·48) guarantees that the term in $p'D$ in (6·26) dominates and the associated Λ is less than zero. Thus the equilibrium is unstable. In the appropriate limit this case corresponds to the complete separation case of § 5.

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