Stably Stratified Turbulence

Consider a flow with a stable mean density stratification,

\[ N^2(z) = \frac{d\bar{\rho}}{dz}(z) > 0. \]  

(1)

Here, as usual, the vertical coordinate is oppositely directed to the gravitational force. An important regime parameter is the gradient Richardson number,

\[ Ri = N^2 / \left( \frac{\partial \bar{u}_h}{\partial z} \right)^2, \]

(2)

where the subscript \( h \) denotes the horizontal component. \( Ri \) measures both the gravitational stability of the stratification and the dynamical competition with the mean shear. When \( Ri < 0 \), the flow will be in a state of convective turbulence for \( Re, Ra \gg 1 \), with possibly some influences by the mean shear if it is strong enough. When \( 0 \leq Ri \leq O(1) \), the flow will be a type shear turbulence (e.g., as in the Kelvin-Helmholtz flow) for \( Re \gg 1 \), with some modification by the weakly stable stratification. However, when \( Ri \gg 1 \), the turbulence is different from either of these regimes, and it is referred to as stratified turbulence when \( Re \gg 1 \). Because the influence of the stratification is strong in this latter regime, its motions are anisotropic, \( w \ll u, v \), because of the large gravitational work required for vertical movement. The behavior of stratified turbulence is in contrast to weakly nonlinear internal wave “turbulence” that also is consistent with \( Ri, Re \gg 1 \) (Sec. 1). In nature away from boundaries, \( Ri \gg 1 \) typically, and both turbulence and wave regimes occur. There remains considerable uncertainty about the proportional occurrences of each type of dynamics. (See 2D Homogeneous Turbulence chapter for the wavenumber spectrum of the atmospheric mesoscale: how much of the observed regime with \( E_h(k) \propto k^{-5/3} \) is due to stratified turbulence and how much to internal waves?)

This regime of \( Ri \gg 1 \) is volumetrically the most common one in the ocean and atmosphere for motions on small scales and outside of planetary boundary layers and convecting clouds. Stratified turbulence does not have an obvious energy source from a local instability of its environment: since \( N^2 > 0 \) the flow is gravitationally stable, and since \( Ri > O(1) \) the flow is Kelvin-Helmholtz stable. Thus, either this regime is one of decaying turbulence or else it must be sustained by an energy cascade from some larger scale phenomenon, such as geostrophic turbulence or inertial and internal gravity waves.

Geostrophic turbulence is most likely to be energetically sustained by the shear instability of large-scale winds and currents, either climatological or transient. Its standard (i.e., quasi-geostrophic) model exhibits an inverse energy cascade, analogous to 2D turbulence, which implies a lack of energization for the stratified regime with \( Fr \ll 1 \) and \( Ro \gg 1 \). Furthermore, there are plausible theoretical arguments why stratified turbulence might itself exhibit an inverse energy cascade (Sec. 1). Nevertheless, there is increasing evidence that forward energy cascades usually occur from larger scales into the stratified turbulence regime and continue down to Kolmogorov universality and microscale dissipation (Sec. 3 and Geostrophic Turbulence).

In most locations inertial or internal gravity waves are energetically sustained by propagation from some remote source, usually located at the vertical boundary and due either to tidal or subtidal flow past topography or to fluctuating boundary stress. Internal waves also can be generated
by convective buoyancy flows, such as gravity currents or cumulus clouds, and even by stably stratified planetary boundary layers. If the waves are strong enough to intermittently break and overturn, then a local, more isotropic cascade to dissipation will occur.

It is an important — and until recently partially open — question whether stratified turbulence satisfies the H1 hypothesis of 3D homogeneous turbulence, i.e., whether \( g \) becomes negligible on sufficiently small scales, but still larger than the dissipation (Kolmogorov) scale. The implication of this not being true is that stratified turbulence would remain anisotropic throughout its cascade to dissipation and — in combination with an expectation of an inverse energy cascade — even have a conundrum about how its dissipation is accomplished. Most previous evidence has favored violation of the H1 hypothesis in situations where stratified advective dynamics is more important than internal wave dynamics; however, most of the present laboratory and computational evidence is from situations where \( Re \) values are not too large (Secs. 1-2). More recent computational evidence supports H1 at large enough \( Re \) (Sec. 3).

1 Internal Waves and Vortical Motions

In terms of characteristic scales, we can write \( Ri \) in terms of the previously introduced Froude number, \( Fr \),

\[
Ri \sim N^2 H^2 / V^2 = Fr^{-2} .
\]  
(3)

(Commonly \( Ri(x) \) is viewed as a local measure of the stratification and shear influences, while \( Fr \) is viewed as a bulk measure.) Thus, stratified turbulence occurs when \( Fr \ll 1 \), and we can develop an asymptotic dynamical approximation as \( Fr \to 0 \) (Lilly, 1983; McWilliams, 1985). We distinguish this regime from geostrophic turbulence (also with \( Fr \ll 1 \)) by neglecting rotational influences, i.e., assuming \( Ro \gg 1 \). Consider a non-dimensionalization of the governing equations by the following advective scales for the stratified turbulence:

\[
\begin{align*}
\mathbf{x} & \sim H, \quad \mathbf{u}_h \sim V, \quad t \sim H/V \\
\phi & \sim V^2, \quad b \sim V^2/H, \quad w \sim V^3/N^2 H^2 .
\end{align*}
\]  
(4)

The resulting non-dimensional Boussinesq equations are

\[
\begin{align*}
\frac{D_h \mathbf{u}_h}{Dt} + Fr^2 w \frac{\partial \mathbf{u}_h}{\partial z} & = -\nabla_h \phi + \nu \nabla^2 \mathbf{u}_h \\
Fr^2 \frac{Dw}{Dt} & = -\frac{\partial \phi}{\partial z} + b \\
\frac{D_h b}{Dt} + Fr^2 w \frac{\partial b}{\partial z} + N^2 w & = \kappa \nabla^2 b \\
\nabla_h \cdot \mathbf{u}_h + Fr^2 \frac{\partial w}{\partial z} & = 0 .
\end{align*}
\]  
(5)

(Here we have retained the diffusive terms, even though we have not made their non-dimensionalization scaling explicit.)

The limiting form for these as \( Fr \to 0 \) is a form of 2D dynamics for \( \mathbf{u}_h = \hat{z} \times \nabla_h \psi \) and \( \phi \), where the flow has only a parametric dependence on \( z \); diagnostic relations for \( b \) and \( w \) from the
second and third equations in (5); and a diagnostic relation for the additional $O(Fr^2)$, horizontally divergent component of the horizontal flow, $u_h = Fr^2 \nabla_h \chi$, from the fourth equation in (5). Here we are using a Helmholtz decomposition of the 3D non-divergent velocity field,

$$
    u = -\partial_y \psi + \partial_x \chi, \quad v = \partial_x \psi + \partial_y \chi, \quad w = -\int^z \left[ \partial^2_x \chi + \partial^2_y \chi \right] dz. \quad (6)
$$

Thus, when $\chi \ll \psi$, the flow is primarily horizontal (as in the example in Turbulent Flows), and it can be called rotational or vortical for its vertical vorticity. In contrast, when $\chi \gg \psi$, the horizontal flow is divergent, and the degree of velocity anisotropy may not be so pronounced.

Thus, to leading order in $Fr$, the dynamics of stratified turbulence is equivalent to the dynamics of 2D turbulence, with an independent evolution of $u_h$ at each vertical level. This is sometimes called layerwise 2D turbulence. Because of sensitive dependence, however, we expect these independent evolutions to diverge, and this should act to decrease the vertical correlation scale and increase vertical shear, with the ultimate effect to somehow couple the evolutions at different levels and bring in some higher-order corrections in (5). One set of approximate equations with such behavior results from carrying the asymptotic analysis through $O(Fr^2)$: it is called the non-rotating Balance Equations, where the horizontal momentum equation in (5) is replaced by approximate forms of its curl and divergence equations (i.e., the vertical vorticity equation and gradient-wind balance), the vertical momentum balance is hydrostatic, and the last two equations in (5) are kept in their entity. The Balance Equations contain vortex-stretching terms that couple the vertical vorticity evolution at different levels, although they have not yet been solved for stratified turbulence and so we cannot yet be sure that they are accurate here, although they are known to be quite accurate for the advective evolution of rotating, stably stratified flows where $Ro = O(1)$. The Balance Equations are also a first-order PDE system. It remains an open question how suitable the Balance Equations are for non-rotating flows for at least three reasons: (1) they are not uniformly valid at $O(Fr^2)$ because they drop the vertical acceleration term in (5) that is formally of this order; (2) there are flow-dependent solvability conditions for their time integrability (McWilliams et al., 1998) that are more likely to be problematic without rotation than with it; and (3) recent solutions at small $Fr$ and large $Re$ indicate that there may be a part of a turbulent Boussinesq solution that exhibits small-scale overturning in a way that is inconsistent with $Ri$ being everywhere large (Sec. 3 below).

Métais and Herring (1989) computationally solved the Boussinesq equations for $Fr(0) \ll 1$ and approximately balanced initial conditions with $\psi \gg \chi$). In the subsequent evolution, $Fr(t)$ remains small and $Fr^4(\nabla_h \chi)^2 \ll (\nabla_h \psi)^2$ continues to hold (Fig. 1, right panel), whereas with more general initial conditions the ratio between $\psi$ and $\chi$ remains $O(1)$ (Fig. 1, left panel). Therefore, we can conclude that the scaling assumptions behind the preceding asymptotic analysis are uniformly valid in time, at least in these solutions and by this global velocity variance measure, hence there is a plausible basis for believing that the dynamics of the Balance Equations may also

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1The third equation contains a time-derivative in it, so it appears to be undetermined as a diagnostic relation for $w$. However, this appearance is illusory, since the time derivative for $b$ cannot advance independently of the time derivative of $\psi$. Thus, the reduced system of (5) with $Fr \to 0$ is a first order PDE system in time. It is often a subtle matter to correctly identify the temporal order of a PDE system. As another example of the subtlety of this assessment, consider that the full Boussinesq system (5) is a third order system, in spite of the occurrence of four time derivatives, and its hydrostatic approximation, after dropping $Dw/Dt$ in the second relation, is also a third order system with three time derivatives.
be accurate. However, the non-rotating Balance Equations develop singularities in certain regions of transition between local vorticity dominance and local strain dominance (McWilliams et al., 1998). Therefore, it must be true that there are some locations where Balanced dynamics break down, perhaps to a locally more isotropic cascade to dissipation or to a local generation of internal wave energy (n.b., the internal wave analysis immediately below with $\chi \gg \psi$). Nevertheless, since a global measure of anisotropy is well preserved under evolution (Fig. 1), it must also be true that these breakdown events are modest in frequency and intensity. Given this seemingly contradictory evidence, as well as that from laboratory experiments (Sec. 2), the theoretical understanding of the advective dynamics of stratified turbulence has been viewed as mysterious in many important aspects.

Note in Fig. 1 that both types of freely evolving turbulence are significantly dissipative. This indicates that a forward energy cascade is occurring — unlike in 2D turbulence — and it suggests an important role for 3D effects on the small scales, either advective or viscous.

We also note that there is another consistent non-dimensionalization for $Fr \ll 1$ that leads to linear internal gravity waves as the leading-order dynamics when $Re \gg 1$:

\[
\begin{align*}
\mathbf{x} & \sim H, \quad \mathbf{u} \sim V, \quad t \sim 1/N \\
\phi & \sim VNH, \quad b \sim VN,
\end{align*}
\]

whence

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \nabla_h \phi - \hat{z}b & = -Fr (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla^2 \mathbf{u} \\
\frac{\partial b}{\partial t} + w & = -Fr (\mathbf{u} \cdot \nabla)b + \kappa \nabla^2 b \\
\nabla_h \cdot \mathbf{u}_h + \frac{\partial w}{\partial z} & = 0.
\end{align*}
\]
Note that the nonlinear terms are of $O(Fr)$ in (8). Yet on an advective time scale — which is $1/Fr$ times the wave time scale in (7) — the nonlinear effects become important, and a weakly nonlinear wave turbulence can induce its own type of cascade to dissipation. Sometimes this dissipation is concentrated in local wave-breaking events that certainly are not weakly nonlinear. In wave breaking events, several outcomes are possible: new gravity waves and/or new stratified turbulence may be generated, and/or a 3D Kolmogorov cascade may be initiated.

Thus, the regime of $Ri \gg 1$ is one where there can be either stratified turbulence or internal gravity waves, or both simultaneously, and the present expectation is that their mutual interactions usually are weak both because their nonlinear coupling terms are estimated to be small and because gravity waves can radiate away from a region, leaving behind a balanced state (i.e., a stratified adjustment process, analogous to geostrophic adjustment when $f$ is significant). In this chapter we will pay more attention to the regime of stratified turbulence.

2 Collapse of Isotropic Turbulence in Stratification

There is a classical problem in stratified turbulence that has been the focus of substantial laboratory experimentation: the evolution of initially isotropic 3D turbulence to its “final” state in a stably stratified fluid. This behavior is also referred to as the “collapse” of “active” turbulence to its “fossil” state\(^2\), although there is an obvious dynamical-regime chauvinism in these terminologies. Consider a situation where a wire mesh is dragged rapidly through a region in a stratified fluid in a way that locally excites isotropic 3D motions; an alternative means of their generation is by flying an airplane or submarine and examining the resulting wake; yet another is to have a local breaking event for a large-amplitude wave in a stably stratified location. I will describe this problem based on a presentation by Tony Maxworthy (1989) at a recent workshop on oceanic turbulence. Prof. Maxworthy is an experimentalist who has worked on the collapse problem for many years.

We define a non-dimensional time,

$$T_* = \frac{Nt}{2\pi}, \quad (9)$$

that we recognize from (4) and (7) as the appropriate non-dimensionalization for gravity waves. Fig. 2 provides a summary of the evolutionary sequence. At early times, $0 \leq T_* \leq 0.5$, the 3D turbulence is modified by the ambient stratification, and the vertical buoyancy flux, $\mathbf{w}b$, is suppressed, after having initially arisen by 3D stirring in the presence of a mean vertical buoyancy gradient. At intermediate times, $0.5 \leq T_* \leq 3.0$, internal waves are generated and propagate away from the turbulent region, and horizontal layers form that are evident in flow visualizations. These horizontal layers do contain anisotropic, advective dynamics, and they expand horizontally to form intrusions into the non-turbulent regions. The vertical scale of these layers when they emerge is approximately the Ozmidov scale, $L_o$, defined by

$$L_o = \sqrt{\frac{\varepsilon}{N^3}}, \quad (10)$$

\(^2\)The term fossil implied the end of advective dynamics with its associated forward energy cascade and dissipation, leaving behind residual density and tracer fluctuations that would slowly diffuse away. We now know that this is a false view, since horizontal motions remain strong and $\varepsilon$ remains large (Fig. 1).
Figure 2: Time line of the evolution of collapsing 3D turbulence in a stratified fluid. (Maxworthy, 1998)
Figure 3: The length scales in collapsing stratified turbulence: Ozmidov scale, $L_O$, integral scale, $L_T$, and Kolmogorov scale, $L_K = \eta$. (Stillinger et al., 1983; Maxworthy, 1998)
and with time they grow in thickness to a scale of $\approx 7L_O$ (sometimes called the Pearson-Linden scale). The Ozmidov scale is the marginal scale where 3D overturning motions of a given intensity, hence cascade and dissipation rate, can occur in the presence of a gravitationally inhibiting stratification: for $L < L_O$ (as is initially true for all the 3D turbulent scales), overturning is not strongly inhibited, and for $L > L_O$, it is. Fig. 3 shows the evolution of the various length scales. As $L_O(t)$ decreases (because the turbulence is decaying and $\varepsilon(t)$ decreases), the turbulent energy peak scale (e.g., the lag-covariance integral scale, $L_T(t)$) grows until $L_T$ becomes larger than $L_O^3$. This continues further until $L_O$ equals the growing Kolmogorov (dissipation) scale,

$$\eta(t) = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$  \hspace{1cm} (11)

at $T_\ast \approx 0.5$, and the vertical buoyancy flux, $\overline{w'b'}$, has decreased to approximately zero (Fig. 4). At late times, $T_\ast \geq 5$, the waves have left the turbulent region, and the remaining turbulence becomes organized into anisotropic coherent vortices, with $w \ll u, v$ and large $|\zeta(z)|$, and a vertical scale like that of the earlier layers. In this stage, the primary contribution to the dissipation rate $\varepsilon$ is through the vertical shear of horizontal velocity,

$$\varepsilon \approx \nu \left(\frac{\partial u_h}{\partial z}\right)^2$$  \hspace{1cm} (12)

(Fig. 5), that is largest at the top and bottom edges of the coherent vortices. There is no indication that $\varepsilon \to 0$ as $Re \to \infty$ in stratified turbulence. Therefore, this cascade and dissipation route is totally unlike that in 2D turbulence, where $\varepsilon$ is due entirely to horizontal velocity shear, $\nabla_h u_h$, and $\varepsilon$ is vanishingly small as $Re \to \infty$. Somehow, though, this vertical kinetic energy cascade coexists with the essentially 2D advective dynamics implicated in the asymptotic scaling analysis in (4)-(5).

This suggests an even simpler approximate model for stratified turbulence with $Fr \ll 1$ than the Balance Equations discussed following (5); viz.,

$$\frac{D_h u_h}{Dt} = -\nabla_h \phi + \nu_h \partial_z^2 u_h \left[ + \nu_h \nabla_h^2 u_h \right]$$  \hspace{1cm} \nabla_h \cdot u_h = 0 . \hspace{1cm} (13)

This is equivalent to viscous 2D layerwise dynamics except for the addition of a dominant vertical diffusion term (a similar proposal was made and discussed by Majda & Grote, 1997). As far as I know the 3D system (13) has never been solved, so it remains an open question whether vertical diffusion suffices to control the growth of variance at small horizontal scales that develops during a forward enstrophy cascade (n.b., this is indicated by the brackets around the $\nu_h$ term in (13), raising the possibility that it may be unnecessary in the presence of $\nu_{v_h}$). Nevertheless, since Fig. 1 shows the evolutionary consistency of the assumption that $Fr \ll 1$, it remains an open question whether the vertical coupling between different layers occurs only viscously, as in (13), or also through the conservative, vortex-stretching dynamics implicit in the Balance Equations. Even if

\footnote{This is a demonstration that there is some degree of inverse energy cascade even in 3D homogeneous turbulence: given energy initially in a narrow wavenumber range, the spectrum breadth will increase to both smaller and larger wavenumbers; the majority of the energy cascades in the forward direction where it is removed by dissipation; this leaves the surviving energy at larger scales on average; hence, $L_T$ increases.}
Figure 4: The vertical heat flux in collapsing stratified turbulence for several different values of $Pr$. (Maxworthy, 1998)
Figure 5: Dissipation in collapsing stratified turbulence over long times. Note that the vertical shear variance accounts for almost all of $\epsilon$. (Maxworthy, 1998)
the latter is the most important mechanism for layer coupling at large $Re$, it may be that (13) is still a useful representation of this effect if we view $\nu_v$ as an eddy viscosity with a larger than molecular magnitude.

![Figure 6: Shadowgraphs of grid-generated turbulence in a stratified fluid, showing the development of density layers when $Nt \gg 1$. (Lin and Pao, 1979)](image)

Some visualizations of the layers and vortices are shown in Figs. 6-8, and some analyses of the horizontal velocity and both vertical and horizontal vorticity fields during the vortex phase are shown in Fig. 9. The latter clearly show the strong shears, $\partial u_h / \partial z$, and the associated large horizontal vorticity. The topology of vortex lines is shown in Fig. 10. Computational solutions for stratified turbulence also show that these *pancake vortices* are the dominant coherent structure for this regime (Métais and Herring, 1989). Pancake vortices are somewhat reminiscent of the axisymmetric vortices of 2D turbulence in plan view, but they are spatially much more closely packed in both the vertical and horizontal directions in stratified turbulence.
Figure 7: Shadowgraph and dye visualization of the wake of a self-propelled slender body in a stratified fluid, viewed from the side. (Lin and Pao, 1979)
Figure 8: Evolution of pancake vortices in the wake of a towed slender body in a stratified fluid, viewed from the top. (Lin and Pao, 1979)

Figure 9: Spatial patterns: horizontal velocity and streamfunction in a horizontal plane (left), horizontal velocity in a vertical plane (upper right), and the orthogonal component of horizontal vorticity in a vertical plane (lower right) in stably stratified turbulence for $Nt \gg 1$. Note the layers of high vertical shear. (Fincham et al., 1996)
Figure 10: Sketches of the pancake vortex shape and vortex line geometry in stably stratified turbulence for $Nt \gg 1$. (Maxworthy, 1998)
3  Equilibrium Stratified Turbulence

Since stratified turbulence, like 3D homogeneous turbulence, is highly dissipative, it must be sustained by some sort of forcing on larger scales. In nature it is not rare for it to occur intermittently, suggesting that its forcing events may be sporadic. As a contrast to the freely decaying and collapsing regimes in Secs. 1-2, now consider an equilibrium regime maintained by random, rotational forcing on scales much larger than $L_0$.

For small $Fr$ and intermediate values of $Re$, the equilibrium behavior is consistent with the decaying behavior: $Ri$ is everywhere small; $\psi$ is $\gg \chi$; pancake vortices are the dominant flow structure; and $\varepsilon$ occurs mainly through the vertical shear variance. What happens as $Re$ is further increased? Does the equilibrium energy level increase as Balanced advective dynamics diminishes the efficiency of the forward energy cascade?

Figure 11: A local overturning event in a simulation of equilibrium stratified turbulence with random rotational forcing at large scales and $Re_\lambda = 1000$. Plotted are instantaneous vertical velocity (greyscale) and $(0, v, w)$ velocity (vectors) in a $(y, z)$ plane; both are normalized by the r.m.s. velocity $V$ indicated by the reference arrow in the upper left. $L$ is the horizontal extent of the domain. (McWilliams, 2004)

Laval et al. (2003) show in computational simulations that there appears to be a boundary in $(Fr, Re)$ space that distinguishes whether or not 2D and Balanced breakdowns occur. When it does it takes the form primarily of local overturning motions in vertically thin layers, where the local $Ri$ becomes small and the local buoyancy gradient may even become negative; the local flow patterns (Fig. 11) are quite similar to free shear layer and Kelvin-Helmholtz instabilities. This boundary is such that for any $Fr \ll 1$, breakdown events never occur for $Re$ below some threshold value $Re_c$, but they do occur, at least infrequently, for larger $Re > Re_c$ (Figs. 12-13). By the requirement that $L_0 \gg \eta$ as $N$ increases for fixed $\varepsilon$, hence that $\nu$ decreases, it seems plausible that $Re_c$ should increase as $Fr$ decreases; e.g., $Re_c \propto Fr^{-2}$ was suggested Riley and deBruynKops, 2003. This is not yet a fully explored behavior, in part because much of the laboratory work on stratified flow and all the previous computational work have been done with rather small values of $Re$.

In summary, for $Fr \ll 1$ there are stratified turbulent solutions to (5) that evolve self-consistently in time (i.e., a $Fr$ based on the outer scales of the turbulence remains small) on a characteristic advective evolution time of $H/V$, with relatively small $w$ compared to $u_h$ and small $\chi$ compared to $\psi$ almost everywhere. Its leading order dynamical approximation is 2D flows in independent
Figure 12: The PDF for local-gradient $\text{Ri}$ values in equilibrium stratified turbulence with $Re_\lambda = 1000$. (An as yet unpublished extension of Laval et al., 2003)

Figure 13: Experimental path in $Re_\lambda(t)$ (black line) for randomly forced stratified turbulence with fixed, small $Fr$ value ($\approx 0.08$) and a step-wise decreasing viscosity $\nu(t)$ every $\Delta \tau = 100$. $\tau$ is a non-dimensional time normalized by the eddy turnover time. Between steps in $\nu$ the flow comes approximately into equilibrium with its forcing. Also shown are time series for the volume fraction of the domain with local $\text{Ri} < 0.25$ (filled grey area) and $\text{Ri} < 0$ (filled black area). There is no occurrence of $\text{Ri} < 0.25$ for $\tau \leq 300$ and $Re_\lambda \leq 500$, but small $\text{Ri}$ values increase with increasing $Re_\lambda$ values $> 500$. (Laval et al., 2003; McWilliams, 2004)
layers, but there are important dynamical corrections involving terms formally of $O(Fr^2)$, many or most of which are probably consistent with the approximate dynamics contained in the Balance Equations. But at sufficiently high $Re$, there also appear to be some local regions with $Ri$ small or even negative, where overturning motions occur and the stratified turbulence cascade transfers energy into “unbalanced motions”, more like internal gravity waves and/or Kolmogorov’s isotropic cascade. Recently Lindborg (2006) has demonstrated a forward energy cascade inertial range, $E(k) \propto \varepsilon^{2/3} k^{-5/3}$, in simulations of equilibrium stratified turbulence with $Re > Re_c$ (Fig. 14)$^4$. This is manifested in vertically layered structures with a local vertical scale, $H \sim V/N$, indicating a breakdown of the asymptotic scaling assumption of $Fr \ll 1$ uniformly in space and time. His solutions show that the flow remains anisotropic even at the largest $Re$ values achieved, although it seems plausible that isotropy will the the eventual outcome on small enough scales, consistent with the validity of the H1 hypothesis (3D Homogeneous Turbulence).

Therefore, while the $k^{-5/3}$ mesoscale wind spectrum near the tropopause could be interpreted as the result of either an inverse or forward energy cascade in stratified turbulence, it now seems most likely that it is a forward range, fed in part by breakdown of geostrophic balance on larger scales (see Geostrophic Turbulence). I.e., it is now not very plausible that an inverse-cascading inertial range, analogous to 2D turbulence, is realizable in stratified turbulence.

Figure 14: Normalized horizontal kinetic energy (solid line) and potential energy (dashed line) in a simulation of equilibrium, randomly forced, stratified turbulence. Note the well developed inertial range with $E \propto k^{-5/3}$ (dotted line). (Lindborg, 2006)

$^4$A companion study (Lindborg, 2005) shows that this behavior persists even in the presence of rotation until the Rossby number $Ro$ drops below a critical $O(1)$ value, where a transition is made to the inverse energy cascade of geostrophic turbulence.
4 Cox-Osborne Model

Geophysical measurements of turbulent velocity and buoyancy fluctuations on very small scales, just larger than $\eta$, provide estimates of the local kinetic energy and buoyancy variance dissipation rates,

$$\varepsilon = \nu (\nabla \mathbf{u})^2 \quad \text{and} \quad \chi = 2\kappa (\nabla b)^2,$$

with the average made over the microscale fluctuations (i.e., on a scale larger than $\eta$). (Note that in this context $\chi$ is the buoyancy variance dissipation rate, not the divergent horizontal velocity potential in (6).) We consider approximate, local equilibrium forms for the TKE and $b^2$ balance equations, neglecting all tendency and transport terms and considering only the turbulent production by the Reynolds stress work associated with the mean vertical shear and the analogous turbulent buoyancy variance generation as the sources balancing the sinks of the dissipation rates:

$$\overline{u_h w'} \cdot \frac{\partial \overline{u_h}}{\partial z} = + \overline{b'w'} - \varepsilon$$

$$\overline{b'w'}N^2 = - \frac{1}{2} \chi.$$ (15)

In these equations, we can insert the eddy-viscosity and eddy-diffusivity definitions in place of the turbulent fluxes,

$$\nu_e = -\overline{u_h w'} / \left( \frac{\partial \overline{u_h}}{\partial z} \right) \quad \text{and} \quad \kappa_e = -\overline{b'w'}/N^2.$$ (16)

This yields a pair of coupled equations for the eddy diffusivities in terms of the measured mean-field gradients and dissipation rates:

$$\nu_e \left( \frac{\partial \overline{u_h}}{\partial z} \right)^2 = \frac{\chi}{2 N^2} + \varepsilon$$

$$\kappa_e N^4 = \frac{1}{2} \chi.$$ (17)

Thus, in a stably stratified fluid with positive eddy transfer coefficients, turbulence generation is entirely by shear production, and it is balanced by the sum of the kinetic energy and buoyancy-variance dissipation rates. For $Fr < 1$, $\varepsilon$ is typically larger than $\chi / N^2$. A common assumption, with considerable empirical support, is that the flux Richardson number,

$$Ri_f \equiv \frac{\overline{b'w'}}{\overline{u_h w'}} \cdot \frac{\partial \overline{u_h}}{\partial z}$$

$$= \frac{\kappa_e N^4}{\nu_e \left( \frac{\partial \overline{u_h}}{\partial z} \right)^2} \quad \left[ \sim 1 / Pr_e Fr^2 \right]$$

$$= \chi / 2 N^2 / (\chi / 2 N^2 + \varepsilon),$$ (18)

has a typical value of about 0.15, hence

$$\frac{\chi}{2} \approx \Gamma \varepsilon \quad \text{and} \quad \kappa_e \approx \Gamma \varepsilon / N^2$$ (19)

for $\Gamma = R_f / (1 - R_f) \approx 0.2$, the so-called mixing efficiency (Toole, 1998). The eddy Prandtl number in (18) is defined analogously to the molecular one, $Pr_e = \nu_e / \kappa_e$. 

18
Another common assumption is that the flow is statistically isotropic near the Kolmogorov scale (i.e., the H1 hypothesis); with this further assumption, then we can estimate $\varepsilon$ and $\chi$ by the following formulas\(^5\):

\[
\varepsilon = \frac{15}{4} \nu \left( \frac{\partial u_h}{\partial z} \right)^2 \quad \text{and} \quad \chi = 6\kappa \left( \frac{\partial T'}{\partial z} \right)^2.
\] (20)

The end result is a procedure for estimating dissipation rates and eddy diffusivities from fine-scale vertical profiles of $b$ (or often just $T$) and $u_h$. It rests on assumptions of local production-dissipation and isotropy that seem solid for shear turbulence with $Fr = O(1)$ and are still somewhat uncertain for $Fr \ll 1$.

5 Oceanic Microstructure and Diapycnal Mixing

The approach in Sec. 4 is often referred to as microstructure estimates of the turbulent transport rates. It was proposed by Osborn and Cox (1972), although there are earlier precedents in engineering turbulence second-moment modeling. It has been used extensively to survey geophysical turbulence, particularly in the upper ocean. An example of this approach is shown in Figs. 15-16 for the upper equatorial ocean. This is a region containing some strong vertical shears associated with the Equatorial Undercurrent. (Another place small $Ri$ can be found is near the Jet Stream, where clear-air turbulence occurs.) The eddy coefficients inferred from (17) are plotted against $Ri$. They are much larger for stratified shear turbulence, with $0 \leq Ri \leq 0.25$ (note that $\tan^{-1}(0.25) = 0.5$), than for more strongly stably stratified turbulence, with $Ri > 0.25$. Even in the latter case, however, the eddy diffusivities are larger than their molecular counterparts by more than an order of magnitude. Note also that $\nu_e > \kappa_e$ in the stratified regime, whereas $\nu_e \approx \kappa_e$ for small $Ri$; i.e., the eddy Prandtl number $Pr_e$ is large for stratified turbulence and $O(1)$ for shear turbulence. Even within the stratified turbulence regime, the eddy diffusivities appear to decrease as $Ri$ increases, albeit only slowly.

Thus, the vertical turbulent fluxes in stably stratified interior regions of the ocean and atmosphere are usually rather small; nevertheless, they can be significant if no other transport process is more efficient. The oceanic pycnocline and the atmospheric stratosphere are two places where this is often so.

Figs. 17-18 are taken from a zonal-vertical section across the Gulf Stream (Fig. 17a) where it passes through the Florida Strait. The stratification (n.b., Fig. 17b, for the potential density, $\sigma_\theta = \rho_\theta - 10^3 \text{ kg m}^{-3}$) is small in the surface boundary layer of less than 50 m (i.e., $\approx 0.5 \text{ MPa}$, in pressure units) thickness, large in the pycnocline reaching down to about 300 m, and again rather small in the deeper water. Because of the strong mean current, $Fr$ (based on density and velocity gradients over $\Delta z = 10 \text{ m}$) in Fig. 17c is nowhere extremely large. Nevertheless, in Fig. 17d $\kappa_b$ from (19) is much smaller in the vertical interior than it is in the top and bottom boundary layers, especially so in the pycnocline. This is also seen in Fig. 18a for profiles averaged over many

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\(^5\)The prefactor for $\varepsilon$ depends upon the distinction between longitudinal and transverse shear variances, with the former half as large as the latter (Batchelor, 1953). The general expression in (14) is comprised of 6 transverse shear variance components and 3 longitudinal ones, implying that $\varepsilon$ proportionality factor for a single transverse shear variance component is $6 + 3/2 = 15/2$ (and half this for the two components in (20).
samples. Very near the bottom (Fig. 18b), the density and current profiles are well mixed over the bottom boundary layer of about 40 m thickness, but the depth interval with $\varepsilon$ much larger than in the interior spans not only the bottom boundary layer but also the 50 m thick stratified shear layer above it.

The most common present interpretation of mixing in the oceanic interior is that internal waves are generated by flow over bottom topography, propagate upwards, and supply mixing energy to a layer of stratified turbulence above the boundary layer, presumably through local “breaking” events (Gregg, 1989). However, the formula (19) for $\kappa_e$ is equally applicable to flows with a flux of energy $\varepsilon$ from larger scales by a forward energy cascade (e.g., stratified turbulence or even geostrophic turbulence if the geostrophic balance constraint breaks down). This predicted form for $\kappa_e$ has recently been confirmed in stratified-turbulence simulations (Brethower and Lindborg, 2008). Thus, as with the open issue about the mix of internal waves and vortical motions in atmospheric and oceanic flows (Sec. 1), so too is the mix of sources for $\varepsilon$ and diapycnal mixing not yet well known.

Fig. 19 shows a compendium of $\kappa_e(z)$ profiles from various locations. Fig. 19a shows what is believed to be the more widespread situation, with the smallest values in the pycnocline and somewhat larger values below, and Fig. 19b shows examples of “hot spots” where the turbulent mixing is unusually high due either to strong local mean shears or topographic features presumed to energize the internal wave field causing local breaking. Overall, the statistics and regime geographies of shear and stratified turbulence in the interior of the atmosphere and ocean are still only partly known.
Figure 16: Eddy diffusion coefficients for momentum, heat, and density as functions of “mean” $Ri$ from the 4.5 day oceanic time series on the equator. For comparison a parameterization by Pacanowski and Philander (1981) is plotted. (Peters et al., 1988)
Figure 17: Measurements of velocity (upper left), potential density (upper right), averaged Froude number computed with 10 m vertical differences (lower left), and eddy diffusivity for density, $K_\rho$ (lower right), across the Florida Strait. The vertical coordinate is pressure, with 1 MPa $\approx$ 100 m depth. (Gregg et al., 1999)
Figure 18: Microstructure profiles in the Florida Strait: through the core of the Gulf Stream (left) and near the bottom boundary layer (right). (Gregg et al., 1999)
Figure 19: Estimates of diapycnal $\kappa_\rho$; (Left) regions not known to be influenced by strong topography or mesoscale currents, and (Right) regions suspected of being mixing hot spots” including fronts, strong mesoscale eddies, islands, seamounts, straits, and canyons. (Gregg, 1998)
References


