Shear Turbulence

Turbulence—especially the types that are found in nature or are of engineering interest—is not spatially homogeneous. Typically it arises in response to instability of some externally-forced circulation with “mean” shear. (In the context of 2D flows, shear instability is called \textit{barotropic instability}.) The usual initial instability occurs at a preferred scale, but subsequently secondary instabilities and non-linear interactions excite other scales of motion and broaden the spectrum following the pathways previous discussed. Of the various types of flow instability, shear is perhaps the most basic. In addition to mean shear instability as a source of turbulence, transient shear is essential to both the cascade and eventual dissipation of turbulence. Shear is also inherent in convective, stratified, boundary-layer and geostrophic flows as well.

In contrast to homogeneous turbulence, shear turbulence is anisotropic due to the preferred direction for the mean flow, and as just noted it usually is inhomogeneous. This means that it will exhibit \textit{eddy–mean interaction}, which is not discussed in the chapters on homogeneous turbulence where it was assumed that there is no mean flow. However, the hypothesis of \textit{Kolmogorov universality} says that on small-enough scales at high-enough \textit{Re}, shear turbulence will be equivalent to 3D homogeneous turbulence.

To illustrate some of the generic properties of shear flow, consider a large-scale or mean flow that varies only in time and with \( z \) plus fluctuations about it, \( \text{viz.} \),

\[
  u_i = U(z,t)\delta_{i1} + u'_i(x,y,z,t). \tag{1}
\]

The coordinate system is oriented so that its \( x \) axis aligns with the mean flow\footnote{The geophysical convention is to have \( z \) as the cross-shear or transverse coordinate, but the engineering convention is to have this be \( y \). Many of the figures used below use the latter convention.}. We can further restrict the case to situations where there is statistical homogeneity in \( x \) and \( y \). In this case over-bars can be viewed either as ensemble averages or represent planar averages over the symmetry coordinates. Note that such a flow is divergence-free and may or may not be developing in time. In terms of vorticity this flow can be described as

\[
  \zeta_i = \frac{\partial U}{\partial z}(z,t)\delta_{2i} + \zeta'_i(x,y,z,t). \tag{2}
\]

For the effect of \( U \) on the dynamics of \( u_i \) to be non-trivial requires the mean flow to be non-uniform, in other words the \textit{shear} must not vanish everywhere:

\[
  S \equiv \frac{\partial U}{\partial z} \neq 0, \quad \text{for some} \ z. \tag{3}
\]

If we assume that the fluid is neutrally stratified, and that rotational effects are negligible (\textit{i.e.}, neither \( f \) or \( g \) are relevant), then the mean flow satisfies an equation of the form,

\[
  \frac{\partial U}{\partial t} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} - \frac{\partial}{\partial x_j} \overline{u'_i u'_j}. \tag{4}
\]

Here \( \overline{u'_i u'_j} \) is the component of the Reynolds stress whose divergence accelerates the mean velocity
in the $x$ direction. The Reynolds stress in turn satisfies an equation of the form,

$$\frac{\partial}{\partial t} u_i' u_k' = -U_j \frac{\partial}{\partial x_j} u_i' u_k' - u_i' u_j' \frac{\partial U_k}{\partial x_j} - \frac{\partial}{\partial x_j} u_i' u_k' u_j' - \frac{1}{\rho_0} \left( u_k' \frac{\partial p'}{\partial x_i} + u_i' \frac{\partial p'}{\partial x_k} \right)$$

$$+ \frac{g}{\theta_0} \left( u_k' \theta_i' \delta_{i3} + u_i' \theta_k' \delta_{i3} \right) + \nu \left( \frac{\partial^2 u_i'}{\partial x_j^2} + \frac{\partial^2 u_k'}{\partial x_j^2} \right), \quad (5)$$

which somewhat simplifies in our situation, because $U_i = U \delta_{i1}$, and it will simplify further with homogeneity in $x$ and $y$.

The energetics of this problem are instructive. For illustration we restrict ourselves to what are called unbounded, or free shear flows. An example of primarily mathematical interest would be that of a triply periodic domain with a fixed pressure gradient, which we refer to as the uniform or homogeneous shear case. Examples of inhomogeneous flows are jets and channel and pipe flows with free slip boundaries where fluctuating quantities vanish. Such flows are sometimes called free-shear layers. For physical realizability they essentially require that $S$ have compact support in $z$, and be unbounded in $x$. Wall-bounded shear flows are also of great interest, but will be treated later in this chapter for some classical examples, and again in the chapter on Boundary-Layer Turbulence.

For the flow (1) with uniform shear, we can derive an equation for the evolution of the mean field energy $E \equiv \frac{1}{2} U^2$:

$$\frac{d\langle E \rangle}{dt} = UF - \langle G \rangle. \quad (6)$$

The force $F$ corresponds, e.g., to the pressure gradient along the flow direction, and it is responsible for accelerating the mean flow. $G$ is the dissipation of mean energy, and for large $Re$ where mean viscous dissipation is negligible, it is equivalent to the generation of fluctuation kinetic energy. It is defined by

$$G = -u'w' \frac{\partial U}{\partial z} = -Su'w'. \quad (7)$$

The evolution of the eddy energy, sometimes call the turbulent kinetic energy (TKE), and denoted by

$$e \equiv \frac{1}{2} u'_i u'_i = \frac{1}{2} \left( u'^2 + v'^2 + w'^2 \right), \quad \text{is given by the trace of } (5) \text{ specialized to } (1):$$

$$\frac{\partial e}{\partial t} + U \frac{\partial e}{\partial x} = -u'w' \frac{\partial U}{\partial z} + \frac{\partial}{\partial x_j} u'_j e' - \frac{1}{\rho_0} \frac{\partial}{\partial x_i} u'_i p' + \nu \frac{\partial^2 e}{\partial x_j^2} - \nu \left( \frac{\partial u'_i}{\partial x_j} \right)^2. \quad (9)$$

The last term on the right-side is the viscous energy dissipation rate, $\varepsilon > 0$. Averaging (9) over the domain and noting that all but the first and last term on the right-side of (9) are in divergence form, and hence vanish under the global integral, results in

$$\frac{d\langle e \rangle}{dt} = \langle G \rangle - \langle \varepsilon \rangle, \quad (10)$$

where we have substituted $\varepsilon$ for the molecular dissipation and $G$ from (7).
The sum of (6) and (10) describes the evolution of the domain-averaged kinetic energy for all scales of motion. It shows that the overall balance is between the mean acceleration associated with $F$ and the dissipation of small scale fluctuations by molecular diffusion. From this perspective $G$ simply measures the rate at which mean kinetic energy is transferred to the fluctuations. To the extent that both the fluctuations and the mean flow are steady in time, it must equal $UF$ and $\varepsilon$. As such it can be thought of as the rate at which turbulence is forced, or fed, into the cascade by the breakdown of the mean shear. On the basis of an eddy-viscosity hypothesis that states that on average fluctuations correlate with the mean shear,

$$u'u'' = -\nu_e \frac{\partial U}{\partial z}. \quad (11)$$

Note that

$$\langle G \rangle = \langle \nu_e S^2 \rangle \geq 0 \quad (12)$$

for any non-negative eddy viscosity field,

$$\nu_e(z,t) \geq 0. \quad (13)$$

On the other hand, if the turbulent fluctuations are uncorrelated with the mean shear (i.e., $\nu_e = 0$), then $\langle G \rangle = 0$, and the turbulence merely decays; however, this is not the equilibrium situation, where $d\langle \varepsilon \rangle/dt = 0$ and $G > 0$ in either an ensemble or time average.

**1 Homogeneous Shear**

From (1) it is apparent that the shear $S$ defines another non-dimensional parameter that is usually defined as

$$Sh = \frac{\sqrt{\varepsilon}}{|S|\ell} \quad (14)$$

where $\sqrt{\varepsilon}$ and $\ell$ are characteristic velocity and length scales of the turbulence. To the extent the turbulence is characterized by a well developed cascade of energy (i.e., the Reynolds number is large), $\ell$ can be related to the dissipation rate similarly to the scaling estimate in 3D homogeneous turbulence:

$$\varepsilon = \frac{e^3}{2\ell}, \quad \text{hence} \quad Sh = \frac{\varepsilon}{|S|\ell}. \quad (15)$$

To the extent the eddy-viscosity hypothesis is valid, the symmetries of the homogeneous shear problem suggest that

$$\nu_e \propto \ell \sqrt{\varepsilon} = \frac{e^2}{\varepsilon} = \frac{e}{Sh |S'|}. \quad (16)$$

In the limit $Sh \rightarrow \infty$, the shear-flow interaction is trivial (i.e., the eddy viscosity and hence $G$ both vanish), and the turbulence behaves as in homogeneous turbulence. In the limit $Sh \rightarrow 0$, the fluctuations behave as in rapid-distortion theory, where the governing equations for $u'$ are linearized about the mean flow — at least until the fluctuation amplitudes grow with $G > 0$, if they do so — to invalidate this approximation. One of the primary effects of rapid distortion is to stretch flow structures in the $\hat{x}$ direction due to differential rates of mean advection $U(z)$ at different heights $z$. After a while this has the effect of tilting and elongating $\zeta_y(x,z)$ in the
downshear direction, which makes the associated \( u'w' \) mostly positive, hence \( G < 0 \); i.e., rapid distortion by strong shear leads to turbulent energy decay even before viscous dissipation acts\(^2\). However, in an equilibrium state (analogous to (22) below), the value of \( Sh \) becomes somewhat less than 1, and the dynamics are distinct from either of the limiting regimes of negligible or strong shear.

Figure 1: History of the component velocity variances normalized by the vector variance in homogeneous shear turbulence. \( u \) is streamwise (longitudinal) velocity, \( v \) is transverse (cross-shear) velocity, and \( w \) is spanwise velocity. Symbols are from data and solid lines are computational results. (Rogers and Moin, 1987)

Equilibrium homogeneous shear turbulence is modestly anisotropic in its velocity field, with

\[
\frac{\bar{u}^2}{q^2} > \frac{\bar{v}^2}{q^2} > \frac{\bar{w}^2}{q^2}.
\]

That is, the downstream (i.e., longitudinal or streamwise) velocity fluctuations are largest, the cross-stream (i.e., spanwise) fluctuations are intermediate in intensity, and the shear-inhibited, cross-shear (i.e., transverse) fluctuations are the weakest. These three components contain, respectively, about 50%, 30%, and 20% of the total turbulent velocity variance (Fig. 1). The eddy-viscosity hypothesis (11) must be true in this situation since turbulence is sustained by \( G > 0 \). Furthermore, the value of \( \nu_e \) in equilibrium must be a \((x,t)\) constant by the symmetries of the problem.

There also are coherent vortices in this turbulent regime. They are called hairpin or horseshoe vortices. They are vortex tubes, as in 3D homogeneous turbulence, but in a shear flow they have a preferred orientation. Dynamically the most significant component of fluctuation vorticity is \( \zeta_y' \), associated with a rotational (i.e., vortical) flow in the \( x-z \) plane, hence with the velocity components \( u' \) and \( w' \) that comprise the mean Reynolds stress in \( G \). \( \zeta_y' \) can grow by stretching the mean vorticity \( S \); on the other hand, because of \( S \), the hairpin vortices do not have to grow from an infinitesimal seed. The stretching strength can be, and usually is, less than in 3D homogeneous turbulence.

\(^2\)This fluctuation stretching and phase tilting is sometimes called Orr’s mechanism.
Figure 2: Computational homogeneous shear turbulence: (a) Distribution of the inclination angle of the projection of the vorticity vectors in the longitudinal-transverse plane in computational homogeneous shear turbulence; (b) Projection of the instantaneous vorticity vectors on a plane inclined at 45° to the mean flow. Note the horseshoe/hairpin structures. (Rogers and Moin, 1987)

Figure 3: Typical vortex lines displaying a hairpin structure in computational homogeneous shear turbulence. (a) 3D view; (b) End view in the spanwise-transverse plane; (c) Side view in streamwise-transverse plane. (Rogers and Moin, 1987)
turbulence (but clearly more than in 2D homogeneous turbulence where it is zero). However, an axisymmetric line vortex along the $\hat{y}$ axis will not contribute to $G$ since the $u'w'$ product will have canceling values in different quadrants of the hodograph so that $\overline{ww'}$ vanishes. Therefore, some distortion from axisymmetry is required, and the right sense for $G > 0$ is elliptical flow lines with elongation in the upper-left and lower-right quadrants in the $x - z$ plane when $S > 0$ (i.e., with $\overline{uw'} < 0$ when $U_z > 0$, such that $G > 0$ in (12, Fig. 2). In terms of vortex lines — that cannot begin or end within a fluid in the absence of body forces — this occurs when the vortex line is distorted away from being parallel to $\hat{y}$ by locally protruding or bulging into the upper-right or lower-left quadrants in local regions where the fluctuation vorticity has become strongest through stretching. A fluctuation vorticity vector pointed (e.g., to the upper-left has $u' < 0$ and $w' > 0$ on its $+y$ side and vice versa on its $-y$ side) both of which contribute to $u'w' < 0$. This shape is qualitatively similar to a horseshoe or a hairpin. Evidence for hairpin vortices is shown in Fig. 3. On the whole, much less is known in detail about uniform shear than about free and wall-bounded shear layers since the latter are so much easier to realize in the laboratory.

2 Free Shear (Mixing) Layer

Figure 4 is an example of a spatially developing free-shear layer. The defining configuration is one where two streams with essentially uniform flow (i.e., $U_1$ and $U_2$) are separated by a “splitter plate” upstream of $x = 0$ and are suddenly put into contact with each other for $x > 0$. For an inviscid flow one could imagine that downstream of the flow the two streams would continue unimpeded, separated only by a semi-infinite sheet of infinite vorticity oriented in the spanwise direction and perpendicular to the transverse plane. Such a configuration is, however, unstable. So fluctuations develop between the two fluid layers, and in the mean they act to smooth out...
the velocity profile and define a zone of finite vorticity. This zone of vorticity and turbulence is observed to thicken in the downstream direction as the fluctuations spread through the respective layers of laminar fluid. Such layers, while typically realized experimentally as shown in Fig. 4, can also develop temporally from a spatially uniform mean shear zone between two streams brought into contact at $t = 0$. Such a configuration is easier to handle computationally (e.g., the streamwise boundary condition can be periodic), and it is consistent with the simple velocity and vorticity representation in (1), whereas spatially developing free-shear layers are easier to realize in the laboratory. In this particular turbulent regime, laboratory experiments performed by engineering fluid dynamicists led the way to discovery, more so than in homogeneous turbulence (even in 3D), and, in particular, it is the context where the term coherent structures first came into common usage.

There is a rough correspondence between the spatially and temporally developing flows, assuming the development scales are “slow” compared to the fluctuation scales and the temporal problem also starts from a velocity step-function like the profile at the tip of the splitter plate. This correspondence is between the statistical properties in the two regimes through the transformation,

$$ x = \frac{1}{2}(U_1 + U_2) t. \quad \text{for } t > 0. $$ (18)

Figure 5: The mean velocity profile $U(\eta)$ at several $x$ (left) and the growth of the mixing region thickness with downstream distance $x$ in an experimental plane mixing layer. (Wygnanski and Fiedler, 1970)

The fluctuations that develop downstream of the splitter-plate take the form of initially small-amplitude, spanwise waves on the shear layer that roll up into large-amplitude, spanwise vortices with $\zeta_y(x, z, t)$ the important fluctuating vorticity component. $\zeta_y$ has the same sign and approximately the same magnitude as the unstable mean shear $S$ (i.e., the initial development is essentially 2D without important vortex stretching). This waves $\rightarrow$ vortices evolution comprises a transition regime that is short in $x$ (i.e., only a few eddy turn-over scales), beyond which the turbulence becomes more fully developed, albeit anisotropic and inhomogeneous. Since the shear and vorticity of the mean flow are confined to the neighborhood of $z = 0$, so is the fluctuation vorticity; in the far field, however, there are substantial fluctuations in pressure and velocity. For the idealized problem with opposing mean flows in the two layers, the problem is symmetric in $z$, although in
real flows the mixing layer reaches slightly farther into the high-speed region\(^3\).

The mixing layer is characteristic of many canonical problems in turbulence: it exhibits extensive symmetry and allows the flow to be described with only a few control parameters. An understanding of the idealized problem is then used to develop concepts that we try to apply more generally. For instance, in the simplest possible spatially developing mixing layer — where viscous, boundary, and density effects are neglected — all we can really ask is how does \( U \) develop as a function of \( x \) and \( z \) and as a function of the imposed velocity scale, \( V = U_1 - U_2 > 0 \). This introduces two fundamental scales (distance and speed) with three variables \((x, z, U)\). It follows that the non-dimensional mean velocity profile must have the following similarity form,

\[
\frac{U(x, z)}{V} = U(\eta) \quad \text{where } \eta \equiv z/x. \tag{19}
\]

(For a review of similarity approaches in fluid dynamics the reader is referred to Appendix B.) This form is a special case of a more general similarity form,

\[
\frac{U(x, z)}{V} = U(\eta, Re) \quad \text{where } Re \equiv V x/\nu, \tag{20}
\]

presuming that the \( Re \) dependence will disappear for a large-enough \( Re \). Physically (19) can be thought of as a statement of the self-similar structure of the velocity profile because all downstream profiles, when suitably averaged and non-dimensionalized, are identical. A corollary to this is that the mixing-layer thickness \( \delta \) must scale with \( x \). Experiments support the idea that the spatially developing layer exhibits Reynolds number similarity and hence is statistically self-similar (i.e., it exhibits scaling behavior) in \( x \) once the turbulence becomes fully developed, with the non-dimensional thickness of the vorticity layer,

\[
\delta/x \approx 1/6, \tag{21}
\]

for the case when \( \rho_1 = \rho_2 \) and \( U_2 = 0 \) (Brown and Roshko, 1974). If we condense the essential “turbulence problem” for 3D homogeneous turbulence to the derivation of the value of the Kolmogorov constant, the analog for the spatially developing mixing layer explains why the non-dimensional thickness is 1/6.

Another property of the mixing layer we would like to explain is the ordering of the velocity fluctuations, \( \overline{u'^2}, \overline{v'^2}, \overline{w'^2} \), whose relative variances are roughly 44%, 31%, 25%, respectively (cf., (17)). The covariances among velocity components do not vanish with this anisotropy (Fig. 8), as expected if momentum is to be transported between the two fluid streams and energy extracted from the mean flow (i.e., \( G > 0 \)). The turbulence is influenced significantly by the mean shear, with an experimental value for \( Sh \) of

\[
Sh = \left( \frac{\max \overline{u'^2}/2}{V} \right) \approx 0.27. \tag{22}
\]

The turbulence is also intermittent (Fig. 7), moderately so in the center of the layer but increasingly so on the periphery, and more so in \( \nabla u' \) than in \( u' \) (i.e., more so on smaller scales). The intermittency can further be expected to increase with \( Re \) as in 3D homogeneous turbulence.

---

\(^3\)In Figs. 5-9, the high-speed side is located at \( \eta < 0 \), whereas in the following figures the high-speed side is on the upper side of the layer.
Figure 6: Transverse (left) and streamwise (middle) velocity fluctuations and Reynolds stress at several $x$ in an experimental plane mixing layer. (Wygnanski and Fiedler, 1970)

Figure 7: Kurtosis profiles for velocity (left) and its streamwise derivative (right) at several $x$ in an experimental plane mixing layer. (Wygnanski and Fiedler, 1970)
The similarity behavior of the solutions implies that the non-dimensional velocity profiles are a function of only one argument, making them amenable to description in terms of ODEs that afford a better chance of deriving analytic results. As we shall see this is a recurrent theme in the treatment of turbulent flows. To illustrate this point, consider the mean continuity balance for \( x > 0 \),

\[
\partial_x U + \partial_z W = 0. \tag{23}
\]

After non-dimensionalizing by \( x \) and \( V \), assuming Reynolds number similarity (19), and using \( \partial_x = -\eta/x \partial_\eta \) and \( \partial_z = 1/x \partial_\eta \), we can express the continuity equation as

\[
-\eta d_\eta U + d_\eta W = 0 \\
\Rightarrow d_\eta W = \eta d_\eta U. \tag{24}
\]

Transverse symmetry requires that \( W = 0 \) at \( \eta = 0 \) hence

\[
W = \int_0^{\eta} \frac{dU}{d\eta} d\eta \\
= \eta U - \int_0^{\eta} U d\eta. \tag{25}
\]

A secondary circulation (i.e., vertical velocity) develops in the mean flow. It increases away from the interface, and its detailed structure is governed by the shape of \( U \).

Next consider the mean streamwise momentum balance, under the assumptions that it is steady (i.e., \( \partial_t = 0 \)), has no streamwise mean pressure gradient (i.e., \( p_x = 0 \)), and is at large enough \( Re \) that viscous terms are negligible:

\[
(u \cdot \nabla)u \cdot \hat{x} = -\nabla \cdot u'(u' \cdot \hat{x}) \\
-\eta \frac{dU}{d\eta} + W \frac{dU}{d\eta} = \eta (u'u')_\eta - (w'w')_\eta \\
\frac{dU}{d\eta} \int_0^{\eta} U(\eta*) d\eta* = -\eta (u'u')_\eta + (w'w')_\eta. \tag{26}
\]

If \( w'w' \) and \( w'w' \) are of comparable size (as they are) and we focus our attention within the core of the mixing layer where \( \eta \ll 1 \) (recalling that \( \delta/x \leq 0.2 \) encompasses the region of strong turbulence), then the preceding relation simplifies to

\[
\bar{w}'w' = \int_{-\infty}^{\eta} d\hat{\eta} \left[ \frac{dU}{d\eta} \int_0^{\hat{\eta}} U(\eta*) d\eta* \right]. \tag{27}
\]

Thus, we can estimate the transverse Reynolds stress from observations of the transverse profile of the mean velocity using (27) and compare them with direct measurements of the Reynolds stress in (Fig. 8). The agreement is fairly good, with observational uncertainties and the importance of
the neglected term, \( \eta (u'w')_\eta \) above, the most likely sources of the discrepancy. Note that, at the center of the layer, the streamwise and transverse velocity fluctuations are fairly well correlated,

\[
\overline{u'w'} \approx -10^{-2} V^2 \approx -0.4 \left( \overline{u'u'w'w'} \right)^{1/2}.
\]

(28)

This indicates that the turbulent fluctuations are fairly efficient at generating Reynolds stress.

Figure 8: Reynolds stress profile at several \( x \) in an experimental plane mixing layer. (Wygnanski and Fiedler, 1970)

We define an eddy viscosity for the transverse turbulent momentum flux by

\[
\nu_e(x, z) = -\overline{u'w'}/U_z \\
= Vx \left[ -\frac{u'w'}{V^2} \right] / U_\eta \\
= Vx \mathcal{N}_e(\eta).
\]

(29)

Measurements show that \( \mathcal{N}_e \) is reasonably constant with \( \eta \) (Fig. 9), even while the eddy flux and mean shear profiles vary substantially. The constant value is such that

\[
\nu_e/Vx \approx 2 \times 10^{-3} \\
\nu_e/V\delta \approx 1.25 \times 10^{-2} \\
Re_e = V\delta/\nu_e \approx 80.
\]

(30)

Notice that the scaling dependence of \( \nu_e \) is velocity X length, as expected in a mixing-length theory (Appendix A) and as commonly found in many types of turbulence. The smallness of the non-dimensional prefactor is also common.
Thus, the “effective” Reynolds number for the flow, based on an eddy viscosity instead of the molecular one, is on the order of a critical $Re$ for a first transition of a laminar flow to instability. We can interpret this as a statement that the turbulence acts on the mean flow to achieve a renormalized, quasi-laminar, mean-field dynamical state, even though the actual Reynolds number based on the molecular viscosity may be very large.

Because the free shear layer was one of the first turbulence regimes to be well explored experimentally, the empirical discovery that a constant eddy viscosity represented rather well the interaction of the turbulence and mean flow had a very strong influence on theoreticians and modelers. From a more modern perspective, the success of such a simple model appears to have been nearly inevitable — only one energy pathway is available here; the developing flow has a high degree of symmetry, leading to the downstream similarity structure; and the turbulent dynamics of the entire layer is tightly coupled by the coherent vortices that span the layer and dominate the flow — in contrast to other, more complicated regimes of turbulence where many pathways seem possible. Nevertheless, this early result did a lot to make the eddy viscosity model the most popular turbulence model, as did its simplicity.

The coherent structures of the free shear layer are spanwise vortices. This shape is the one that arises from the instability of the vortex sheet that separates the two layers, and it can be related to the hairpin vortices of uniform shear (Sec. 1), the vortex tubes of 3D homogeneous turbulence, and the axisymmetric monopole (line) vortices of 2D homogeneous turbulence. The coherent vortices of the free shear layer have a smaller projection of $\zeta'$ onto the perpendicular directions, $\hat{x}$ and $\hat{z}$, than in the former two regimes (i.e., they are more constrained in their alignment to be mainly spanwise), but certainly more than in the 2D regime (where the projection is zero). Because the vortices approximately fill the width of the layer, they do not have the spatial freedom to make
larger deformations in \((x,z)\) that would extend out of it. Thus, we expect the vortex dynamics of the free shear layer to be more like those of 2D turbulence than 3D, at least compared to uniform shear.

Figure 10: Shadowgraphs of an experimental plane mixing layer in a uniform density fluid. Simultaneous plan (top) and edge views (bottom). (Roshko, 1992)

Figure 11: Instability, vortex formation, and pairing (merger) in an experimental free shear layer. (Roshko, 1992, adapted from Freymuth’s figure)

We demonstrate the nature of these vortices pictorially in Figs. 10-12. A brief summary of their contents is the following:

- An experimental visualization of the coherent structures (vortices) in the \((x,z)\) plane.
- A depiction from experiments of the \((x,t)\) trajectories of individual vortices, showing pairing (merger) and size growth.
Figure 12: Vortex emergence and evolution for a computational 2D parallel-flow shear layer with finite but small viscosity and tracer diffusivity. The two columns are for vorticity (left) and tracer (right), and the rows are successive times: near initialization (top); during the nearly linear, Kelvin-Helmholtz, varicose-mode, instability phase (middle); and after emergence of coherent anticyclonic vortices and approximately one cycle of pairing and merging of neighboring vortices (bottom). (Lesieur, 1995)
• Numerical solutions in 2D of the temporally developing mixing layer, showing the initial instability, vortex pairing, and symmetry breaking (i.e., sensitive dependence).

• An experimental visualization of the same behaviors.

• The PDF of spacings between coherent vortices from experiments, grouped under the similarity hypothesis.

• A sketch, based upon experiments, of the streamwise vortices (a.k.a. braids) that emerge as a secondary instability of the coherent spanwise vortices.

• Numerical solutions in 3D of the temporally developing mixing layer, showing the braids.

It has also been shown that the turbulent intensity and mixing rates can be controlled by harmonic excitations near the splitter plate, with either enhancement or suppression depending upon whether the excitation reinforces or cancels the instability process by acting in or out of phase with the linearly unstable eigenmode at its eigenfrequency.

2.1 Turbulence Modeling

As a final topic in this section, we now will consider several alternatives for turbulence modeling, i.e., replacing all or part of the turbulent fluctuations in an equation for the mean or resolved flow by parameterizing the dynamical effects of the deleted scales of motion on the retained ones.

\[ \frac{D\mathbf{u}}{Dt} = -\nabla \phi + \nabla \cdot \tau, \]  

(31)

where \( \phi \equiv p/\rho_0 \) and \( \tau \) is the stress tensor,

\[ \tau_{ij} = -u'_i u'_j + \nu S_{ij}, \]  

(32)

with \( S \) is the rate-of-strain tensor,

\[ S_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}. \]  

(33)

For constant \( \nu \) (as in a Newtonian fluid), \( \nabla \cdot \nu S = \nu \nabla^2 \mathbf{u} \), using continuity.

Consider the following collection of models for \( \tau \):

**Molecular viscosity:** \( \tau = \nu S \) (i.e., no turbulence). This, of course, fails as a turbulence model as soon as \( Re \) is large enough that the laminar shear flow becomes unstable and turbulence develops; this can be calculated in a Direct Numerical Simulation (DNS).

**Eddy viscosity:** \( \tau = \nu_e S \), for some non-negative \( \nu_e \sim VL \gg \nu \). Figure 9 above indicates this would be a fairly successful model for calculating the mean flow in the free shear layer if the right magnitude were chosen for a spatially uniform \( \nu_e \). In general, however, such a model is intrinsically \( ad \ hoc \), with respect to the magnitude and spatial distribution of \( \nu_e \), and lacks any \( a \ priori \) guarantee that it will be accurate. Nevertheless, it is a commonly used parameterization in
Figure 13: Trajectories of vortices in an experimental plane mixing layer. (a) Crosses denote identified locations of spanwise vortices in the \((x, t)\) plane. (b) Sequence continuing from (a) but showing trajectories as continuous lines. Note the merger events and growth of vortex size. (Brown and Roshko, 1974)
Figure 14: Distribution of spacings between coherent spanwise vortices in a turbulent free shear layer. $\bar{\lambda}$ is the mean spacing. Vertical bars are from an experiment, and the curve is a log-normal distribution. (Roshko, 1992, adapted from Bernal’s figure)

Figure 15: Sketch of the topology of streamwise vortex lines in a plane mixing layer. (Bernal and Roshko, 1986)
Figure 16: Top view of the low-pressure (top) and high-vorticity (bottom) fields in a 3D Large Eddy Simulation of an incompressible temporal mixing layer. The mean shear flow is from left to right. (Lesieur, 1995)
Large-Eddy Simulations (LES) that partially resolve the turbulent fluctuations. One common form for $\nu_e$, called the TKE model, is

$$\nu_e = c_t e^{1/2} \lambda,$$  \hspace{1cm} (34)

where $e$ is the TKE for the subgrid-scale (SGS) motions, $\lambda$ is an eddy length scale, and $c_t$ is a constant of order unity. Such a form, however, is not complete until $e$ and $\lambda$ are specified, and this is often accomplished in a moment-closure hierarchy with independent equations for $e$ and $\lambda$ (usually, but not always, closed at the level of second moments, the variances and covariances). Many variants of this type of moment-closure model exist and are used in practical applications. One class is the $k - \epsilon$ model (where $k$ is the same as $e$), and its eddy viscosity is $\nu_e \propto e^2 \epsilon^{-1}$).

For a pathway into this large literature, see Lumley (1990) and the more recent oceanic parameterization scheme, the General Ocean Turbulence Model (GOTM) (Umlaff and Burkhard, 2005: http://www.gotm.net/).

A simpler choice is called the Smagorinsky model, with

$$\nu_e = c_s^2 (\Delta x)^2 ||S||,$$  \hspace{1cm} (35)

where $\Delta x$ is the grid size in the LES model, $||S|| = S_{ij} S_{ij}$ (with summation over repeated indices), and $c_s$ is called the Smagorinsky constant. The rationale for this formula is an assumption that the associated energy dissipation rate derived from (31), $\nu_e ||S||^2$ is locally and instantaneously equal to the the dissipation rate at the Kolmogorov scale (identified with $\Delta x$) in 3D homogeneous turbulence, viz., $\epsilon = V^3/L = \nu_e V^2/L^2$ $\Rightarrow \nu_e = V L = |\nabla u| (\Delta x)^2$. This argument yields (35), with the added tunable constant $c_s$. Thus, the Smagorinsky model presumes the universality of 3D homogeneous turbulence on small scales at large $Re$, without any consideration for intermittency. For the free shear layer it would be a poor model choice for the mean flow alone (Fig. 9), but it is reasonably successful in LES calculations when the dominant coherent vortices are resolved. (35) is a mean- and/or resolved-field closure, and thus it is much simpler to implement than (34) and its accompanying TKE equation.

Eddy viscosity with stochastic backscatter:

$$\tau_{ij} = \nu_e S_{ij} + \varepsilon_{ijk} \Psi_k$$
$$\partial_j \tau_{ij} = \partial_j(\nu_e S_{ij}) + a_i,$$  \hspace{1cm} (36)

where $\nu_e$ is specified from (35); $\varepsilon_{ijk}$ is the alternating tensor (i.e., $= 1$ if $(i,j,k)$ is an even permutation of $(1,2,3)$, $= -1$ if an odd permutation, and $= 0$ otherwise); and

$$a = \nabla \times \Psi$$  \hspace{1cm} (37)

is a random, non-divergent acceleration. Here $\nu_e$ is chosen by considerations such as those discussed above; its role is to accomplish the necessary dissipation, on average, that is required for $G > 0$ in the (implicit) SGS TKE budget. The role of $a$ is to add natural variability (and perhaps even intermittency) to the resolved flow, even in the mean-field limit where there are no resolved turbulent motions. A simple formula for $a$ is

$$\Psi_k = c_a ||S||^{3/2} (\Delta t)^{-1/2} (\Delta x)^2 r_k,$$  \hspace{1cm} (38)
where \( r \) is a Gaussian random variable with zero mean and unit variance with independent realizations in each coordinate direction and at each \((x, t)\) in the computational grid (Leith, 1990). Figure 17 shows that stochastic backscatter provides a plausible mean-field model for the free shear layer, using (36)-(38).

### 3  Jets and Wakes

A classical problem in engineering turbulence is a jet at large \( Re \), although this is not a flow configuration that often occurs in the ocean and atmosphere; for this reason we will not cover this topic here. Loosely speaking, jet turbulence has a lot in common with free shear layer turbulence, albeit with some important configurational differences, most notably a mean shear field that is not merely of one sign. Similarly a turbulent wake, once it detaches behind its generating obstacle is also somewhat like a free shear layer with mean shear of both signs. In contrast to jets, wakes commonly occur in the ocean and atmosphere behind topographic obstacles.

### 4  Shear Boundary Layers

Turbulent flows near solid boundaries are also layers with finite thickness \( \delta \), just like the free shear layer. These two shear layers types have many properties in common, although there are also some important differences due to the wall boundary condition, \( u = 0 \) at \( z = 0 \). We will only briefly discuss the shear boundary layer, in part because of its similarities with the free shear layer (e.g., its coherent structures are hairpin vortices that “burst” away from the boundary into the interior).
Figure 18: Sketch of a jet flow. Here the Jet is symmetric (uniform) in the spanwise direction. The jet is characterized by its mass flux or momentum impulse $M$, as well as $Re$. Inset shows a turbulent jet from a rocket on a test stand. (Roshko, 1992)
and in part because we will discuss the geophysical generalization of this regime, the planetary boundary layer (PBL), more extensively in later lectures.

Consider two classical examples of shear boundary layers. The boundary layer over a flat plate in an otherwise unbounded domain is inherently a developing layer, either spatially or temporally; its thickness will grow either downstream or in time without limit unless it is arrested by the finite domain size (this is in contrast to the PBL where stable stratification and rotation can provide a late-time limit on $\delta$). The cause of this boundary layer is a finite mean horizontal velocity in the fluid interior, $U_\infty$. Another example is fully developed turbulence in a pipe (i.e., Poiseuille flow) or a channel, driven by a nearly uniform, mean downstream pressure gradient $\partial_x P$ and an associated peak velocity $U_c$ in the center of the pipe. Here the boundary layer thickness grows until it is comparable to the pipe diameter $d$ and thereafter it is no longer developing and is in an equilibrium state independent of $x$. A related boundary layer that can equilibrate its thickness is Couette flow, i.e., the flow between two flat plates moving with different velocities.

In equilibrium pipe, channel, or Couette flow, we can decompose the domain into three regions (Fig. 19-20): the viscous sublayer near the boundary, where diffusion must become of $O(1)$ dynamical importance as $u \to 0$, hence a local $Re \to 0$, approaching the wall; the near-axis region (or, more generally, the outer region) in the pipe center where the boundary layers meet from all radial directions; and the intermediate region or surface layer in between, where the local $Re$ is large and only the distance $z (\ll d)$ to the nearest boundary is significant. We define the friction velocity by

$$u_* = \sqrt{\tau/\rho_o},$$

where $\tau$ is the mean tangential stress on the boundary ($= \nu \partial_z \overline{u}(z = 0)$) and $\rho_o$ is the fluid density.

In the classical simple theory based on dimensional analysis, it is assumed there is a surface layer close to the boundary but outside of any direct influences of viscosity that might occur very close to the boundary. In this case we assume all quantities of interest can depend only on the wall stress and the distance from the boundary, which is presumed to limit the size of the dynamically dominant eddies. In particular, the only dimensionally correct dependence for the mean shear on $u_*$ and $z$ is the following one:

$$\partial_z \overline{u}(z) = C u_* z,$$

where $C$ is a constant. This can be integrated to give the famous law of the wall, with its logarithmic dependence on distance,

$$\overline{u}(z) = \frac{u_*}{k} \log \left[ \frac{z}{z_o} \right],$$

where $C = 1/k$ has been rewritten in terms of von Karman’s constant, $k \approx 0.41$ (not to be confused here with wavenumber), and $z_o$ is an integration constant called the roughness length and interpreted as characterizing the small-scale irregularity of the underlying wall boundary (note that $z_o$ is irrelevant here when $z \gg z_o$).

More generally a boundary layer profile may also depend on the viscosity $\nu$ and the outer scales of the flow, the speed $\max[\overline{u}]$, and the boundary layer thickness $d$. By the procedures of dimensional analysis, we can expect that there is a functional relationship among all the independent non-dimensional parameters that can be constructed from the relevant dimensional quantities. Thus, we can define the non-dimensional streamwise velocity profile and tangential distance by
Figure 19: Profiles of the total stress, Reynolds stress, and viscous stress across a boundary layer ($Re_\delta = 7 \times 10^4$). Note 30-fold change in abscissa scale at $y/\delta = 0.05$ (Tritton, 1988)

Figure 20: Example of a turbulent boundary layer mean velocity profile plotted on log-linear coordinates for $Re = 2 \times 10^4$. The abscissa is non-dimensional distance from the boundary in “wall units”, i.e., $\xi = z u_*/\nu$. Solid lines correspond to viscous sub-layer and log layer theoretical expectations, and broken lines show data fits. (Tritton, 1988)
the boundary stress and a viscous length scale, \( \nu/u_* \), as follows:

\[
\phi = u/u_*, \quad \eta = zu_*/\nu.
\]

We further construct an outer-scale Reynolds number by \( Re = \max[\pi]d/\nu \). Thus, the more general form of (40) above is

\[
\frac{\partial \phi}{\partial \eta} = \frac{1}{\eta} \Phi(\eta, Re),
\]

where \( \Phi \) is a general functional of its arguments, yet to be specified. Note that (40) is recovered if we assume that, for \( Re \gg 1 \) and \( \eta \gg 1 \),

\[
\Phi \rightarrow \Phi(\infty, \infty) = \frac{1}{k},
\]

then the mean velocity profile has the form

\[
\bar{\phi} = \frac{1}{k} \log \eta + \text{const},
\]

\[
\bar{u}(z) = \frac{u_*}{k} \log \left[ \frac{z}{z_o} \right].
\]

The physical interpretation of (44) is that once the distance from the boundary is large enough in “viscous units”, \( \nu/u_* \), and once the Reynolds number is large enough, then the dependences on each of these non-dimensional quantities disappears (cf., (49)-(50) below).

Associated with the logarithmic profile is a turbulent Reynolds stress that matches the boundary stress,

\[
\bar{u}'w' = u_*^2 + O(z),
\]

and a diagnostic eddy viscosity (i.e., as in (11)),

\[
\nu_e = u_* z + O(z^2).
\]

Thus, \( \nu_e \) is certainly not spatially uniform (unlike in the free shear layer), and it approaches zero moving towards the boundary. Since the mean shear does not approach zero, the eddy viscosity in the Smagorinsky model (35) is inapplicable; instead a smooth transition can be made to an analogous surface layer model based on the mean shear, where

\[
\nu_e = c_m^2 z^2 ||S||.
\]

This has a surface-layer dependence \( \nu_e \sim z \), as in (47) since \( S = u_*/kz \) in (28). If we define \( S' = S - \bar{S} \), we can distinguish the regimes of validity of (35) and (48) by the degree that \( ||S'|| \) or \( ||\bar{S}|| \) dominates \( ||S|| \), viz., outside of and within the surface layer respectively. SGS parameterizations based on a combination of (35) and (48) perform well in LES models of the shear PBL (Sullivan et al., 1994). Outside of the surface layer, \( \nu_e \) is no longer proportional to \( z \).

The assumption (44) need not be accurate, however, for large but finite \( \eta \) and \( Re \). As an alternative, Barenblatt et al., 1997) have argued that a power-law form,

\[
\Phi(\eta, Re) = A(Re)\eta^{\alpha(Re)},
\]
leading to
\[
\bar{\phi} = \left( A_0 + \frac{A_1}{\log(Re)} \right) \eta^{\alpha_1/\log(Re)}
\] (50)
asymptotically for \(Re \gg 1\), provides a better fit to the experimental data than does (42)-(45). The issue is presently unresolved, since the distinction between these two profile forms can be quite subtle in measurements at achievable values of \(Re\). On the other hand, for large enough \(Re\) values — that sometimes occur in the ocean and atmosphere — the distinction can be quantitatively significant. Our opinion is that the conventional wisdom, (44) rather than (49), is to be preferred by Ockham’s razor: it is not empirically discredited, and it is simpler. Most shear flow experts probably share this view.

The coherent structures of a shear boundary layer are also called horseshoe or hairpin vortices (as in homogeneous shear; Sec. 1). In a boundary layer, they are localized to originate in the surface layer that has the highest mean shear and develop (or burst, as commonly stated) into the outer region of the boundary layer (Figs. 21- 22). As in the homogeneous regime, their particular orientation in the streamwise-transverse plane is associated with the requisite Reynolds stress (i.e., \(u'w' < 0\)) and energy conversion (\(G > 0\)).

![Figure 21: Sketch of a hairpin or horseshoe vortex in a turbulent boundary layer. (Tritton, 1988)](image)

A canonical wall-bounded shear flow is plane Couette flow. Here the mean flow \(U(z)\) is forced by the differential motion of two solid parallel boundaries. There is a steady, viscous, laminar solution available in this geometry, \(\text{viz.}, U \propto z\); in this case there is a uniform tangential viscous stress, \(\nu U_z\), that carries the momentum from the equal and opposite wall stresses on either side. However, for large enough \(Re\), turbulence can become fully developed, and the equilibrium mean
Figure 22: Visualization of turbulent boundary layer in planes 45° to the flow direction. (a)-(b) $Re \approx 5 \times 10^3$; (c)-(d) $Re \approx 8 \times 10^4$. (Tritton, 1988)
profile can have its shear collapse into relatively thin boundary layers near the walls (Fig. 23). Even in this case the total transverse stress, \( -\overline{u'w'} + \nu U_z \), must be independent of \( z \) in equilibrium; if this is represented as eddy diffusion, \( \nu_e U_z \), then \( \nu_e(z) \geq 0 \) must increase away from the walls (as in the surface layer form (47) above) to a larger value in the central region with little local \( z \) dependence in either \( U \) or \( \nu_e \). In particular, unlike in the free shear layer, this requires that the turbulence extend vigorously throughout the domain to sustain the Reynolds stress even where the mean shear is weak. Perhaps not surprisingly, then, turbulent Couette flow also has domain-filing coherent vortices that are very large in size compared to boundary-layer hairpin vortices (Komminaho et al., 1996; Papavassiliou & Hanratty, 1997). As in other shear flows, the coherent vortices are the agents of the mean Reynolds stress.

Figure 23: Comparison of laminar and turbulent boundary layer velocity profiles. Laminar: Blasius profile. Turbulent: experimental profile with \( Re_\delta = 10^6 \). (Tritton, 1988)

An oddity of plane Couette flow is that the profile \( U \propto z \) has no linear instability (i.e., no inflection point where \( U_z \) changes sign, as required in a Rayleigh integral theorem), no matter how large \( Re \) becomes; thus, here the transition to turbulence requires finite-amplitude instability for the growth of fluctuations to their equilibrium levels. A related problem is the Taylor-Couette flow between two concentric, rotating cylinders; often its behavior is very similar to that of plane Couette flow, though for some ratios of the cylinder rotation rates a linear centrifugal (also called symmetric or inertial) instability can arise for \( Re \) above a critical value.

4.1 Laminar Boundary Layer Theory

Although atmospheric boundary layers are often turbulent, we begin our study with a review of classical laminar boundary layer theory. This theory proves useful in shaping our thinking about boundary layers in general and helps set the stage for our later discussions. We focus on two examples that help illustrate the basic idea of what a boundary layer is, why it arises, and what the source of its name is. The first example is Blasius’s boundary layer that develops over a flat plate; the second is Ekman’s solution for a steady-state boundary layer in a rotating fluid.
4.1.1 Flat Plate Boundary Layer

Imagine a free flow that suddenly impinges on an infinitesimally thin flat plate, as depicted in Fig. 24. The flow going over the flat plate “notices” the presence of the plate in so far as its velocity must vanish at the surface.

To understand how the flow might respond, we analyze the incompressible 2D Navier-Stokes equations with constant density:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{align*}
\]

subject to the boundary conditions

\[
u = w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad u = U \quad \text{at} \quad z = \infty.
\]

Here \( U \) is the undisturbed interior fluid velocity.

Recall from the previous chapters that in order for the viscous term to be important at otherwise large \( Re \) (thereby allowing the lower boundary conditions to be satisfied) a new length-scale, associated with viscosity and the no-slip boundary condition, must emerge. This length-scale, denoted by \( \delta \), should scale with \((\nu \tau)^{1/2}\) where \( \tau \) is an advective timescale. At large \( Re \) we expect \( \delta/L = \epsilon << 1 \) where \( L \) is an advective length-scale. Thus admitting a role for viscosity introduces a second length-scale into our problem. This length-scale is expected to be relevant to velocity derivatives perpendicular to the plate, while the advective length-scale \( L \) should be relevant to stream-wise velocity derivatives, such that

\[
\frac{\partial}{\partial z} = \frac{1}{\epsilon L} \frac{\partial}{\partial \tilde{z}} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial \tilde{x}},
\]

where tilde denotes a non-dimensional quantity.

Applying (55) to the continuity equation and using the outer velocity \( U \) to scale \( u \) implies that \( w \) must scale with \( \epsilon U \) if the streamwise velocity divergence is not to vanish at leading order. In
addition, if we assume that the pressure gradient in the boundary layer is that “impressed” by the outer flow, then because the outer-flow must satisfy:

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x}, \]  

(56)

the pressure density ratio must scale with the velocity squared

\[ \frac{P\tilde{\rho}}{P\tilde{\rho}} \propto U^2. \]  

(57)

Based on this scaling our governing equations may be written non-dimensionally as follows:

\[ \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \]  

(58)

\[ \frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{w} \frac{\partial \tilde{u}}{\partial \tilde{z}} = -\frac{1}{\tilde{\rho}} \frac{\partial \tilde{P}}{\partial \tilde{x}} + \frac{1}{Re} \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{\epsilon^2} \frac{\partial^2 \tilde{w}}{\partial \tilde{u}^2} \right) \]  

(59)

\[ \epsilon \frac{\partial \tilde{w}}{\partial \tilde{t}} + \epsilon \tilde{u} \frac{\partial \tilde{w}}{\partial \tilde{x}} + \epsilon \tilde{w} \frac{\partial \tilde{w}}{\partial \tilde{z}} = -\frac{\epsilon}{\tilde{\rho}} \frac{\partial \tilde{P}}{\partial \tilde{z}} + \frac{1}{Re} \left( \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + \frac{1}{\epsilon} \frac{\partial^2 \tilde{w}}{\partial \tilde{z}^2} \right), \]  

(60)

where the scaling of the \( \partial P/\partial z \) term has been determined by the scaling of all of the other terms in (60).

Equation (59) reaffirms our earlier arguments, viz., if viscosity is to have a role it must be in a layer whose relative scale \( \epsilon \) scales with \( Re^{-1/2} \). Under this scaling (58)-(60) imply

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \]  

(61)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \]  

(62)

at leading order. These are Prandtl’s boundary layer equations. They can be written as a single equation in terms of the streamfunction

\[ \psi(x, z) \quad \text{where} \quad u = \frac{\partial \psi}{\partial z}, \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x}. \]  

(63)

Prandtl’s student, Blasius, solved these equations for the special case of steady flow with no mean pressure gradient. In terms of the streamfunction Blasius’s equation is

\[ \frac{\partial \psi}{\partial z} \left( \frac{\partial^2 \psi}{\partial x \partial z} \right) - \frac{\partial \psi}{\partial x} \left( \frac{\partial^2 \psi}{\partial z^2} \right) = \nu \frac{\partial^3 \psi}{\partial z^3}. \]  

(64)

In addition to the \( x \) (the distance from the plates leading edge) and \( z \) (the height above the plate) both the free stream velocity \( U \) and \( \nu \) might be expected to control the nature of the solution; i.e., we might expect solutions of the form,

\[ \psi = g(x, z, \nu, U). \]  

(65)
Based on the dimensions of the parameters of the problem, the Π theorem (Appendix B) suggests that the non-dimensional stream function should be a function of no more than two dimensionless numbers. Here we choose them to be the free stream Reynolds number $Ux/ν$ and $z/x$, and we non-dimensionalize $ψ$ by the free-stream velocity and length scales $U$ and $x$:

$$ψ = xU \tilde{g}(z/x, Re). \quad (66)$$

If we look for solutions exhibiting incomplete similarity in $Re$ (Appendix B), viz that

$$ψ = \frac{xU}{Re^\alpha} f \left( \frac{z}{x} Re^\alpha \right), \quad (67)$$

we find that solutions satisfying (64) do exist for $\alpha = 1/2$. This means that

$$ψ = (νxU)^{1/2} f(\eta) \quad \text{where} \quad \eta = z \left( \frac{U}{νx} \right)^{1/2}, \quad (68)$$

should be a solution of (64). Instead of looking for a solution in terms of a function $ψ(x, z)$ (or $f(z/x, Re)$) of two independent variables, our problem becomes one of solving for a function $f(\eta)$ of the single independent variable $\eta$. By reducing the dimensional order of our system by one, we go from solving a PDE to an ODE. The fact that the problem exhibits incomplete similarity in $Re$ leads to a drastic simplification; after substituting (68) into (64), the problem reduces to

$$f(\eta) \frac{d^2f}{d\eta^2} + 2 \frac{d^3f}{d\eta^3} = 0, \quad (69)$$

subject to the boundary conditions

$$f(0) = \left. \frac{df}{d\eta} \right|_{\eta=0} = 0 \quad \text{and} \quad \left. \frac{df}{d\eta} \right|_{\eta=∞} = 1. \quad (70)$$

Solutions to (70) can be obtained by series methods and are plotted in Fig. 25, where excellent agreement with theory is found. Because the flow asymptotically adjusts to the outer flow, the boundary layer thickness is not an unambiguous quantity. If we equate the boundary layer thickness $h$ with the point where $u \approx 0.99U$, we find that this corresponds to $\eta \approx 5$,

$$h \approx 5 \left( \frac{νx}{U} \right)^{1/2}. \quad (71)$$

Alternatively we can measure the thickness of the boundary layer in terms of integral scales. One such scale is the displacement height,

$$δ_1 \equiv \frac{1}{U} \int_0^∞ [U - u(z)] \, dz \approx 1.721 \left( \frac{νx}{U} \right)^{1/2}, \quad (72)$$

another is the momentum thickness,

$$δ_2 \equiv \frac{1}{U^2} \int_0^∞ u(z) [U - u(z)] \, dz \approx 0.664 \left( \frac{νx}{U} \right)^{1/2}. \quad (73)$$
Figure 25: Velocity distribution in a laminar boundary layer on a flat plate at zero interior flow incidence angle. (Adapted from Schlichting, 1987, Fig. 7.9)

All of these measures of boundary layer depth differ in their precise values of the $O(1)$ proportionality constant multiplying $\sqrt{\nu x/U}$, but they are otherwise similar.

The Blasius solutions can be integrated to yield the net viscous drag on the flow per unit spanwise distance and over some downstream distance $x$ of flow over a plate:

$$D = \int_0^x \nu \left( \frac{\partial u}{\partial z} \right)_{z=0} \, dx = f''(0) \nu U \left( \frac{U}{\nu} \right)^{1/2} \approx \nu^{1/2} U^{3/2} \frac{1}{3x^{1/2}}. \quad (74)$$

A remarkable feature of this solution is that the drag scales with $U^{3/2}$ rather than $U$ as would be expected in the absence of a boundary layer$^4$. This ability of boundary layers to qualitatively change the drag regime of a flow is one of their most important effects.

Blasius’s problem illustrates how a laminar boundary layer, associated with a viscous length scale, develops in the flow. It also shows that this boundary layer develops self-similarly, meaning that at each point downstream of the plate’s leading edge, the flow is non-dimensionally equivalent to the flow at every other point. The interesting twist in this solution is that rather than the solution being scaled by what one might expect in the absence of a boundary layer (i.e., $U$), the appropriate velocity scale is actually a combination of both the external velocity scale and a diffusive velocity scale, (i.e., $(U \nu/x)^{1/2}$). The Blasius problem neatly illustrates that the concept of a boundary layer is intrinsically bound up with the development of an additional length-scale in the flow, in this case one associated with viscous effects. It also provides a powerful illustration of more advanced similarity concepts.

$^4$In more turbulent regimes and ones dominated by pressure form stress, the total drag of an obstacle on the passing flow typically scales as $U^2$ and the associated proportionality constant is called the drag coefficient $C_D$. 

31
4.1.2 Ekman Layer

Another laminar boundary layer of some interest is the Ekman layer. For an Ekman layer we are concerned with how viscous effects and a no-slip lower boundary manifest themselves in the presence of rotation. The framework we take is a rotating plane with the outer-flow in geostrophic balance such that

\[
-f V_g = -\frac{1}{\rho} \frac{\partial P}{\partial x},
\]

\[
f U_g = -\frac{1}{\rho} \frac{\partial P}{\partial y}.
\]  

(75)  

(76)

For this flow we look at \( u(z) \) and \( v(z) \) for a subset of Prandtl’s equations on a rotating plane:

\[
-\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},
\]

\[
-f v = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2},
\]

\[
f u = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2},
\]  

(77)  

(78)  

(79)

subject to the boundary conditions

\[(u(0), v(0)) = (0, 0) \quad \text{and} \quad (u(\infty), v(\infty)) = (U_g, V_g). \]  

(80)

Here \((-fv, fu)\) is the apparent Coriolis force associated with our non-inertial reference frame. In addition to adding the \( f \) terms, these equations differ from the original Prandtl equations because time-derivatives and advective terms are neglected; furthermore the inclusion of \( f \) forces us to consider the third spatial dimension.

If we restrict ourselves to problems wherein \( d^2 U_g/dz^2 = d^2 V_g/dz^2 = 0 \), then we substitute for the pressure gradients using (75) and (76); this allows us to rewrite (78) and (79) in terms of the velocity defects \((u - U_g)\) and \((v - V_g)\) :

\[
\nu \frac{d^2}{dz^2} (u - U_g) + f (v - V_g) = 0
\]

\[
\nu \frac{d^2}{dz^2} (v - V_g) - f (u - U_g) = 0.
\]  

(81)  

(82)

Or, equivalently

\[
\left[ \frac{d^2}{dz^2} - i \frac{f}{\nu} \right] \phi = 0 \quad \text{where} \quad \phi = (u - U_g) + i(v - V_g),
\]

(83)

obtained by first multiplying (82) by \( i \) and then adding it to (81). Solutions take the form

\[
\phi = \phi_+ \exp \left( \frac{i f}{\nu} \frac{1}{2} \right) + \phi_- \exp \left[ - \left( \frac{i f}{\nu} \right)^{1/2} z \right].
\]  

(84)
If we note that

$$i = e^{i \pi/2} \implies \sqrt{i} = e^{i \pi/4} = \frac{1}{\sqrt{2}} (1 + i),$$  \hspace{1cm} (85)$$

it becomes apparent that to satisfy the boundary conditions at \( z = \infty \) requires \( \phi_+ = 0 \) while satisfaction of the boundary conditions at \( z = 0 \) requires that \( \phi_+ = -U_g - iV_g \). Thus by taking the real and imaginary parts of \( \phi \), we obtain solutions for \( u \) and \( v \):

$$u = U_g \left[ 1 - e^{-\kappa z} \cos(\kappa z) \right] - V_g e^{-\kappa z} \sin(\kappa z) \hspace{1cm} (86)$$

$$v = V_g \left[ 1 - e^{-\kappa z} \cos(\kappa z) \right] + U_g e^{-\kappa z} \sin(\kappa z), \hspace{1cm} (87)$$

where \( \kappa = \sqrt{f/2\nu} \) in the northern hemisphere (\( f > 0 \)).

Solutions for (86) and (87) for \( V_g = 0 \) are plotted in Fig. 26. As was the case for the Blasius problem, they show that the action of viscosity leads to the development of a new length-scale in the flow. This length-scale defines the Ekman boundary layer. As we did for the Blasius problem we can define the boundary layer depth in terms of the displacement thickness. For the case where our coordinate system is aligned with the geostrophic wind (so that \( V_g \) vanishes)

$$\delta_1 \equiv \frac{1}{U_g} \int_0^\infty [U_g - u] dz = \int_0^\infty e^{-\kappa z} \cos(\kappa z) \ dz = \left( \frac{\nu}{2f} \right)^{1/2}. \hspace{1cm} (88)$$

Alternatively some define the boundary layer thickness \( h \) to be the level where \( (u, v) \) is aligned with \( (U_g, V_g) \), whence \( h = \pi \sqrt{2\nu/f} \). In either case we find that the depth of the boundary layer increases with viscosity and decreases with rotation.

Other things to notice about the Ekman solution is that the wind at the surface is rotated \( \pi/4 \) radians relative to the geostrophic wind so that in a coordinate system aligned with the geostrophic

![Figure 26: Ekman boundary layer solutions for \( U_g = G \). In the left panel we plot the \( u \) and \( v \) winds. In the right panel we plot the hodograph [the parameterized curve \( u(z), v(z) \) in the \( u-v \) plane], this illustrates the famous Ekman spiral.](image-url)
wind the surface wind is has an equal magnitude in both the $x$ and $y$ directions. Unlike the Blasius solutions the drag at the surface of the Ekman boundary layer

$$\nu \left( \frac{du}{dz} + \frac{dv}{dz} \right) = \left[ (U_g^2 + V_g^2) \frac{8\nu^3}{f} \right]^{1/2}$$  \hspace{1cm} (89)$$
scales with the magnitude of the geostrophic wind, and the three-halves power of viscosity. Because the Ekman boundary layer allows steady solutions the drag at the surface must be balanced by pressure gradients that accelerate the flow within the boundary layer. These pressure gradients act on the unbalanced (sub/super-geostrophic) component of the flow, i.e., the velocity defect. Physically we can think of surface drag decelerating the flow to the point where the velocity defects become large enough such that the unbalanced part of the pressure gradient balances the drag. As rotation weakens, the acceleration is increasingly less effective and thus must occur over an increasingly deep layer. In the limit as $f \to 0$, $\delta \to \infty$. Here steady solutions are no longer possible, analogous to $\delta(x)$ growing without bound as $x \to \infty$ in Prandtl’s problem.

There is, of course, a turbulent analog to the laminar Ekman layer that we will not discuss in detail (n.b., McWilliams, 2006, Chap. 6). Its principal departure from the laminar mean velocity profile with constant $\nu$ is that the turbulent eddy viscosity profile, $\nu_e(z)$, is parabolic, or convex, in its shape (largest in the middle of the layer, approaching zero at the surface, as necessary in a surface layer, and also decreasing while approaching the interior edge), so that the surface current is not as strongly rotated relative to the surface stress and the Ekman spiral is less tightly wrapped.

### 4.1.3 Secondary Circulations

Both the Ekman and the Blasius boundary layers have residual vertical circulations. For the Blasius boundary layer the convergence of the stream-wise wind drives a large-scale circulation away from the plate:

$$w = \frac{1}{2} \left( \frac{\nu U}{x} \right)^{1/2} [\eta f' - f] .$$  \hspace{1cm} (90)$$

In contrast the Ekman boundary layer only has an implied vertical circulation when the forcing varies spatially. By continuity, and the requirement that the undisturbed (outer) flow is divergence free,

$$w(z) = \int_0^z \left( \frac{\partial V_g}{\partial x} - \frac{\partial U_g}{\partial y} \right) e^{-\kappa z} \sin(\kappa z) dz ,$$  \hspace{1cm} (91)$$

where we note that the term in the brackets is just the geostrophic vorticity $\zeta_g$. In the case where $\zeta_g$ is constant with height, it can be taken outside of the integral so that

$$w(z) = \zeta_g \int_0^z e^{-\kappa z} \sin(\kappa z) dz, = \delta \zeta_g .$$  \hspace{1cm} (92)$$

This indicates that the vertical motion is positive for cyclonic ($f\zeta_g > 0$) motion and negative for anti-cyclones. Physically we can see the effect of the boundary layer is to turn the flow in the direction of the pressure gradient. This leads to cross isobaric flow that is divergent in high pressure regions and convergent in low pressure regions.

These residual circulations are important because they are thought to interact with the large scale flow. For instance in the Ekman boundary layer, it is the residual circulation that organizes
the tea leaves into the center of the cup upon stirring. It is also this residual circulation that is responsible for stretching or contracting vortex tubes, thereby either spinning up or spinning down a large-scale circulation on a timescale much faster than the viscous timescale $L/\nu^2$. The effects of residual circulations in atmospheric analogs to Ekman flows have also been suggested as a mechanism for organizing convection in the atmosphere, and through this process have been implicated in a large-scale instability called CISK (Convective Instability of the Second Kind).

5 Kelvin-Helmholtz Turbulence

Although some scientists use this term broadly for shear turbulence away from boundaries, we will reserve it with the narrower meaning of the instability and turbulence of a free shear layer with a stabilizing vertical density gradient. It is an idealized flow configuration relevant to the prevalent regime of stratified turbulence in nature). Its governing parameters are $Re$ and the Richardson number,

$$Ri(z) = \frac{N^2}{(\partial_z U)^2},$$  \hspace{1cm} (93)

that represents the ratio between the stabilizing mean density stratification ($N^2 > 0$) and the destabilizing mean velocity shear variance. Small $Ri$ values are conducive to instability and turbulence, while large values do not generate and support turbulence. Linear stability analyses have implicated a critical Richardson number value of $Ri_c = 0.25$ (Miles and Howard, 1961). The experimental results for Kelvin-Helmholtz turbulence indicate that the turbulent intensity and the vertical heat and momentum eddy fluxes increase rapidly as $Ri$ drops below $Ri_c$.

We use several figures from a computational study of freely decaying Kelvin-Helmholtz turbulence by Werne and Fritts (1999) to illustrate the following points:

- Fig. 27: An initially unstable, stratified shear layer develops layer-sized overturning motions (sometimes called cat’s eye vortices) that are structurally similar to the spanwise vortices of the unstratified mixing layer (Sec. 2). After mixing the turbulent layer collapses under the ultimately dominant influence of the stable stratification.

- Fig. 28: The turbulent intensity grows and decays.

- Fig. 29: The vertical eddy heat flux grows and decays, while the value of $Ri$ increases above $Ri_c$.

- Fig. 30: The vertical and horizontal wavenumber spectra approximately achieve Kolmogorov’s universal form during the period of peak turbulence.

Readings

Session 1 in Whither Turbulence (Lumley, 1990).
Figure 27: Vorticity magnitude in a freely decaying stratified shear layer exhibiting a developing Kelvin-Helmholtz instability. Only a third of the vertical domain is shown. (Werne and Fritts, 1999)
Figure 28: Fluctuation kinetic energy (TKE) and vorticity component maxima vs. time in a freely decaying stratified shear layer exhibiting Kelvin-Helmholtz instability. (Werne and Fritts, 1999)

Figure 29: Horizontally averaged (a) Heat flux and (b) $\text{Ri}$ at the same times as in Fig. 27. The dashed line at the final time in (b) indicates $\langle \text{Ri} \rangle = 0.25$. (Werne and Fritts, 1999)
Figure 30: Horizontal spatial spectra during intense turbulence for temperature and velocity. The upper/lower set of curves show spectra vs. \( h\kappa_h \) and \( h\kappa_v \). Smooth curves show Kolmogorov’s predicted \( k^{-5/3} \). (The \( h\kappa_v \) spectra are shifted down by \( 10^{-6} \).) (Werne and Fritts, 1999)
References


Appendix A: Mixing Length Theory

Recall that for turbulent flows we are interested in the behavior of the mean horizontal wind,

$$\frac{\partial U}{\partial t} + U_j \frac{\partial U}{\partial x_j} - V f = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + \nu \nabla^2 U - \frac{\partial}{\partial x_j} \langle u' u'_j \rangle,$$

(94)

where here we use uppercase to denote the average quantities, so that $U \equiv \langle u \rangle$. The form of this equation differs from the starting point for the Ekman and Blasius equations by the presence of the Reynolds stress. Rewriting (94) using Prandtl’s boundary layer assumptions (Sec. 4.1) leads to

$$\frac{\partial U}{\partial t} + U_j \frac{\partial U}{\partial x_j} - V f = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + \nu \nabla^2 U - \frac{\partial}{\partial z} \langle u' w' \rangle.$$

(95)

Further analysis of this system requires some representation for the Reynolds stress. The most common, and simplest is given by Prandtl’s mixing length theory.

Prandtl hypothesized that $\langle u' w' \rangle$ could, despite arising from advective interactions, behave diffusively. To see the cause consider a purely zonal mean flow as depicted in Fig. 31. If $u'(z_1)$ is associated with fluid being transported bodily a distance $\ell$ up or down the velocity gradient, we expect eddies of scale $\ell$ to mix fluid with the properties of the flow at $z_1 + \ell$ or $z_1 - \ell$ to the level $z_1$. From the Taylor’s series expansion of the flow, the momentum at these locations is just

$$u(z_1 + \ell) = u(z_1) + \ell \frac{dU}{dz} + O(\ell^2) \implies u'(z_1) = \ell \frac{dU}{dz} + O(\ell^2).$$

(96)

By continuity $u'$ and $w'$ should be the same order so that

$$\langle u' w' \rangle \approx -\ell^2 \left| \frac{dU}{dz} \right| \frac{dU}{dz} + O(\ell^3).$$

(97)

Noting that $\ell^2 \left| dU/dz \right|$ has units of diffusivity allows us to write

$$\langle u' w' \rangle = -K_e \frac{dU}{dz}.$$

(98)

This is Prandtl’s mixing-length hypothesis. The term $K_e$ is a property of the flow because it depends both on the assumption of a mixing length $\ell$ and the velocity gradients. It has the units of diffusivity.
Appendix B: Similarity (by B. Stevens)

A powerful conceptual framework for understanding many aspects of turbulent structure is something called similarity theory. A similarity approach to problem solving a problem can be exemplified by considering a pendulum.

Suppose we know nothing about the laws of classical mechanics, yet want to derive an equation for the period, \( \tau \), of a pendulum. Imagine conjecturing that the only essential elements of a pendulum are the fact that it consists of a mass \( m \) suspended by a string of some length \( l \) fixed at some point in space in a gravitational field \( g \). This ideal pendulum is illustrated in Fig. 32. Of course real pendulums involve many other parameters: the size and shape of the mass; material properties of the string such as its charge, surface structure, or elasticity; and the properties of the fluid through which the pendulum moves. But somehow these are all deemed to be non-essential to the dynamical idea of a pendulum, which is as much a mathematical construct as it is a physical one.

If we accept that the only essential parameters of a real pendulum are those that coincide with the ideal pendulum, then we are forced to conclude on dimensional grounds that the period of a pendulum is given such that

\[
\tau = \alpha \sqrt{\frac{l}{g}},
\]

where \( \alpha \) is a constant of proportionality. We could go further and say that because the ideal pendulum is universal (all ideal pendulums are similar) the proportionality constant must also be universal. This hypothesis could then be tested by measuring \( \alpha \) while varying \( l \) and \( g \) using a set of pendulums that come as close to the ideal as possible — for instance by suspending very dense, but small masses from a thin wire in a vacuum. To the extent that \( \alpha \) is a constant, we would gain confidence in the theory. Moreover, once the theory for the ideal pendulum has been developed,
we could systematically attempt to extend it to account for the variety of other effects inherent in real pendulums.

We could take a decidedly less Platonic view of the pendulum by attacking the problem in the presence of other parameters. For the sake of argument, assume that the viscosity \( \nu \) of the medium in which the pendulum moves, and perhaps the diameter \( d \) of the wire, are critical parameters. If we admit these two parameters, we might imagine that

\[
\tau = f(g, l, \nu, d) .
\]  

(100)

We can nondimensionalize this system using \( g, l, \) and \( m \) as our fundamental scales. Such a procedure implies a relation between a nondimensional period \( \hat{\tau} = \tau/\sqrt{l/g} \) the nondimensional viscosity \( \hat{\nu} = \nu/\sqrt{l^3 g} \) and \( \hat{d} = d/l \) the nondimensional diameter,

\[
\hat{\tau} = F(\hat{\nu}, \hat{d}) .
\]  

(101)

In the case where

\[
\lim_{\hat{d} \to 0} \left[ \lim_{\hat{\nu} \to 0} F(\hat{\nu}, \hat{d}) \right] = \alpha ,
\]  

(102)

we recover our additional hypothesis, but from a slightly different vantage point. That is (99) can be seen as following from the assumption that \( F \) approaches \( \alpha \) for sufficiently large \( \hat{d} \) and \( \hat{\nu} \). As long as these two nondimensional parameters are small enough, we can view \( F \) as essentially indistinguishable from \( \alpha \).

Above we equated the idea that \( \alpha \) is universal with the idea that all ideal pendulums are similar. What we meant by this is illustrated by noting that physically the constant \( \alpha \) is the nondimensional period. That is, if we nondimensionalize \( \tau \) by the physical scales of the problem, we arrive at \( \tau = \alpha \). What this says is that all nondimensional pendulums have the same period, and thus are similar.

**B.1 Memories of Second Grade**

This idea of nondimensional equivalence (equivalence of period in the above example) is what we mean when we speak of similarity. It perhaps can be made more clear by considering an example that is familiar to everyone. The idea of similar triangles. Two triangles are considered to be similar if they are nondimensionally equivalent. If we take triangle \( A \) in Fig. 33 and nondimensionalize it by its longest length, and compare it to triangle \( B \) nondimensionalized in a similar fashion we will find that the nondimensional versions of \( A \) and \( B \) are equivalent.

This is not true of triangles \( B \) and \( C \) or \( A \) and \( C \). We could have arrived at this conclusion without actually nondimensionalizing our family of triangles by examining their nondimensional parameters, i.e., their angles. Thus we speak of triangles being geometrically similar if they are equivalent in their nondimensional parameters (their angles) as this implies that they will be equivalent upon nondimensionalization. Hence, our association of similarity with the idea of nondimensional equivalence.

**B.2 The \( \Pi \) Theorem**

Clearly similarity is related to dimensional analysis. This relation is made explicitly clear through something called the \( \Pi \) theorem, where \( \Pi \) denotes a nondimensional parameter. The idea of the
Figure 33: Similar Triangles. Triangles A and B are similar because after nondimensionalization (so that their longest sides are unity) they \((A'\text{ and } B'\text{ above})\) are identical.

Theorem is as follows. In solving any physical problem we are often interested in a quantity we will call \(A\) as a function of the known parameters of the problem. That is, we are searching for a functional relationship:

\[ A = f(a_1, \ldots, a_n, b_1, \ldots, b_k), \quad (103) \]

where \(\{a_1, \ldots, a_n, b_1, \ldots, b_k\}\) are \(n + k\) governing parameters or independent variables. Here we distinguish between the \(n\) variables \(a_i\) whose dimensions are independent of one another, and the \(k\) parameters \(b_j\) whose dimensions can be expressed in terms of the fundamental dimensions defined by the \(a_i\). In the non-platonic pendulum example, the parameters \(l, g,\) and \(m\) would correspond to \(a_1, a_2,\) and \(a_3,\) and, because they can be nondimensionalized by suitable combinations of \(l\) and \(g,\) the parameters \(d\) and \(\nu\) would correspond to \(b_1\) and \(b_2.\)

Mathematically we speak of the \(b_j\) representing the largest subset of parameters that can be nondimensionalized by the remaining independent variables in the problem. It follows that

\[ [b_j] = [a_1]^{k_1j} [a_2]^{k_2j} \cdots [a_n]^{kn_j}, \quad (104) \]

where \([x]\) denotes the dimensionality of \(x.\) Similarly, if \(A\) is to be a function of the given quantities, its dimension must be expressible in terms of these quantities:

\[ [A] = [a_1]^{k_1} [a_2]^{k_2} \cdots [a_n]^{kn}. \quad (105) \]

These results can be used to introduce the \(k + 1\) nondimensional parameters,

\[ \Pi_j = \frac{b_j}{a_1^{k_{1j}} \cdots a_n^{kn_j}}, \quad (106) \]

\[ \Pi = \frac{A}{a_1^{k_1} \cdots a_n^{kn}}. \quad (107) \]
From (103) we find that

$$\Pi = \frac{f(a_1, \ldots, a_n, b_1, \ldots, b_k)}{a_1^{k_1} \ldots a_n^{k_n}} = \frac{f(a_1, \ldots, a_n, \Pi_1^{k_1}, \ldots, \Pi_k^{k_k})}{a_1^{k_1} \ldots a_n^{k_n}}$$

However, because these equations describe a physical relationship in so far as \( A \) is scaled by a given independent variable, (108) must be independent of the unit system chosen. It must be invariant under contractions or dilations of the independent units of the system, which implies that

$$\mathfrak{F}(a_1, \ldots, a_n, \Pi_1, \ldots, \Pi_k) = \mathfrak{F}(\Pi_1, \ldots, \Pi_k).$$

In the language of our pendulum example, the nondimensional period of the pendulum, \( \alpha = \tilde{\tau} \), should be independent of our choice of unit system.

Barenblatt (1996) calls this the central result of dimensional analysis. It is often referred to as Edgar Buckingham’s \( \Pi \)-theorem. It states that any physical relationship between \( n \) quantities of independent dimensions and \( k \) quantities with dependent dimensions can be expressed in terms of a functional relationship in \( k \) nondimensional parameters. The great reduction in governing parameters is that it often greatly simplifies our search for relationships among variables in systems with many degrees of freedom. For instance for the case of the Platonic pendulum, the use of the \( \Pi \) theorem tells us that the period of the nondimensional pendulum is not a function of two variables, \( l \) and \( g \), but rather a universal constant.

### B.3 Similarity Solutions

To illustrate the power of the method, we review a few further examples whereby useful results can be obtained by the similarity method. These should extend our familiarity with the method, and better illustrate its power. The first and most famous example is due to Stuart Turner who predicted the strength of a nuclear explosion using time-lapse photographs and similarity arguments. The second example is from bounded shear flows. It provides a nice introduction to more refined ideas in similarity theory and is also a good stepping stone to the material in the following chapter.

From a distant perspective an atomic explosion can be viewed as an instantaneous release of energy at a point. This release of energy causes a shock wave that propagates outward from the source at some rate. We can ask whether a relationship between the radius \( r \) of the shock wave, the time \( t \) since the explosion, and the energy \( E \) of the explosion can be determined. That is, we seek a function \( f \),

$$r = f(E, t, x_1, \ldots, x_n),$$

where \( \{x_1, \ldots, x_n\} \) are other, possibly important, parameters not yet considered. By hypothesizing that the only other relevant variable is the density of the medium through which the shockwave travels, we are led to the conclusion that the nondimensional radius,

$$\Pi = \frac{r}{E^{1/5} t^{2/5} \rho^{-1/5}};$$
must be universal.

Turner used this argument along with time-lapse photographs of American nuclear explosions and some insights from gas dynamics (indicating that for a similarity theory following (113), \( \Pi \) should be \( O(1) \)) to deduce the energy of the explosions. As told by Barenblatt, Turner’s deductions were rather embarrassing to the U. S. Government; it had classified this information as top secret even though it had released the photographs to the public.

Another important and interesting use of similarity arguments is in the deduction of the velocity profile in a wall-bounded turbulent shear flow. Here we could postulate that the only relevant parameters in the problem are the drag on the lower boundary \( \tau \), the density of the fluid \( \rho \), and the distance from the wall \( z \). Given the associated dimensions,

\[
[
\tau\] = \text{kg m}^{-1} \text{s}^{-2} \quad [
\rho\] = \text{kg m}^{-3} \quad \text{and} \quad [z] = \text{m},
\]

all are independent, and the nondimensional velocity gradient should be universal. That is, if we search for a relationship

\[
\frac{du}{dz} = f(\tau, \rho, z),
\]

then dimensional arguments should lead us to conclude that the nondimensional velocity gradient,

\[
\Pi = \frac{du}{dz} \left( \frac{z \rho^{1/2}}{\tau^{1/2}} \right),
\]

must be universal. The above relationship is experimentally well supported, and the universal value of \( \Pi \) is denoted \( 1/k \) where \( k \) is named for von Kármán who proposed its existence using slightly more circuitous arguments. The quantity \( \sqrt{\tau/\rho} \) has units of velocity and is the friction velocity \( u_* \) in (39). In terms of these quantities our similarity results suggest that

\[
\Pi = \left( \frac{z}{u_*} \right) \frac{du}{dz} = k^{-1} \quad \Rightarrow \quad u(z) - u(z_0) = k^{-1} u_* \ln(z/z_0).
\]

This latter relation is called the law of the wall (41), an important paradigm for turbulent boundary layers.

### B.4 Kinds of Similarity

This law of the wall illustrates some lesser known aspects of similarity approaches to solving problems. For instance, in this problem we could have postulated that the viscosity is an important parameter, or that some external lengthscale \( \Lambda \) associated with the depth of the channel or height of the boundary layer also played a role. If we had done so, all that we could have concluded on dimensional grounds would be that

\[
\Pi = \left( \frac{z}{u_*} \right) \frac{du}{dz} = \mathcal{F}(\Pi_1, \Pi_2),
\]

where \( \Pi_1 \) is effectively an inverse Reynolds number, \( \Pi_1 = Re^{-1} = \nu/(u_* z) \), and \( \Pi_2 = \Lambda/z \). Here we see that our original result is recovered when

\[
\lim_{\Pi_1 \to 0} \left[ \lim_{\Pi_2 \to 0} \mathcal{F}(\Pi_1, \Pi_2) \right] = k.
\]
In this case we speak of *complete similarity* in the parameters $\Pi_1$ and $\Pi_2$. Such phrasing is in accord with the idea of Reynolds number similarity, wherein the dependence of a flow on $Re$ is hypothesized to vanish for sufficiently large $Re$.

In the absence of complete similarity we often find the case of what Barenblatt calls *incomplete similarity* or *similarity of the second kind*. In this case

$$\lim_{\Pi_1 \to 0} \mathcal{F}(\Pi_1, \Pi_2) = \Pi_1^\alpha \mathcal{G}(\Pi_2/\Pi_1^\beta).$$

(120)

This limit implies that

$$\bar{\Pi} = \mathcal{G}(\bar{\Pi}_2),$$

(121)

where

$$\bar{\Pi} = \frac{\Pi}{\Pi_1^\alpha} \quad \text{and} \quad \bar{\Pi}_2 = \frac{\Pi_2}{\Pi_1^\beta}.$$  

(122)

The existence of such a limit leads to a considerably reduction in the complexity of our system. Physically the rescaling that takes place in going from $\Pi$ to $\bar{\Pi}$ reflects the fact that there exists some scale such that our problem can be reduced to an functional relationship in one parameter, but that this scale is not determined by a dimensional analysis. In the case of the shear flow problem, it says that there is a timescale, such that the flow depends only on $\Pi_2$, but this timescale is a product of a viscous timescale $z^2/\nu$ and an inertial timescale $z/u_\ast$. The proposal by Barenblat et al., 1997, discussed in Sec. 4, is an incomplete-similarity embellishment of the complete similarity form in (117).

We raise this issue here because it is often encountered. Some flows do not necessarily tend to a well defined limit as $Re^{-1}$ or $Ra^{-1} \to 0$. Instead some of their important characteristics may retain a power-law dependence on $Ra$ or $Re$ (e.g., intermittency measures).

### B.5 Similarity Thinking and Dimensional Analysis

Too often similarity thinking is confused with straightforward dimensional analysis, and thereby trivialized to a certain extent. We note here that the approach is considerably more sophisticated. Overall it can be thought of as a three step process that blends insight, dimensional analysis, and empiricism.

Insight is perhaps the key ingredient. After the fact it is often all to obvious what the critical ingredients or parameters are in a problem. However, when faced with new problems, we often consider a confounding list of possibly important parameters. Sorting the chaff from the wheat, the essential from the non-essential, is never easy; it depends on sound scientific judgment, experience, intuition, and perhaps a little luck. Once a critical list of parameters has been proposed, the dimensional analysis follows in a straightforward fashion, although there can be better or worse choices for the of independent scales. The last step is empiricism. In the examples above this comes down to measuring an empirical constant. But in other more complicated examples it involves evaluating a similarity function of the form $\mathcal{F}(\Pi_1)$. Typically if these functions depend on more than one nondimensional variable, their experimental determination becomes prohibitively difficult, and as a result most of the similarity theories we deal with will either have $\mathcal{F}$ as constant or be a function of a single parameter.
Similarity Exercises

1. Even for platonic pendulums the similarity argument we developed for the nondimensional period neglects a nondimensional number which, if we did not already know something about classical mechanics, could be important. What is it?

2. For the shear flow problem discussed in Sec. 5, solve for the velocity assuming incomplete similarity (with power-law exponent $\alpha$) in $\Pi_1$ and complete similarity in $\Pi_2$.

3. Suppose for the shear-flow problem of Sec. 5 we use similarity arguments to search directly for a nondimensional form for the velocity. What would the answer look like and what measurements would be necessary to evaluate $u/u_*$ as precisely as if we solved the similarity problem for the nondimensional shear?

4. Suppose you were interested in the stream-wise pressure drop $dp/dx$ in a pipe. Pose this as a similarity problem, identifying important parameters and possible nondimensional dependencies. That is, derive a procedure for solving this problem semi-empirically, assuming that you know nothing about the laws of fluid motion.