The Heating of the Solar Corona

II. A Model Based on Energy Balance

R. W. P. McWhirter
Appleton Laboratory, Astrophysics Research Division, Abingdon

P. C. Thonemann
Department of Physics, University College of Swansea, Singleton Park

R. Wilson
Department of Physics and Astronomy, University College London

Received June 28, 1974, revised January 21, 1975

Summary. The density and temperature distribution of the solar corona is calculated assuming an energy balance between thermal conduction and radiated power loss with the primary heating of the corona by the dissipation of sound-waves propagated upwards from below the sun's surface. A sharp transition region is found and the calculated results are compared with observations. A detailed model atmosphere for the transition region and corona is derived using the Harvard Smithsonian Reference Atmosphere (for the chromosphere) as starting point.

Hydrostatic equilibrium is assumed in the calculations but it is also shown that a pressure arises because of the sound waves which is of comparable magnitude to hydrostatic pressure. The inclusion of this pressure introduces difficulties that are discussed.

Key words: solar corona model — heating mechanism — energy balance — acoustic waves

1. Introduction

Biermann (1948) and Schwartzschild (1948) first suggested that the heating of the sun's corona might be explained by acoustic waves, generated in the convection zone, being transmitted into the sun's upper layers where their energy is dissipated as heat. Kuperus (1969) has reviewed the subject and more recent papers relevant to the present investigations are Athay (1969), Ulmschneider (1970, 1971), Kopp (1968, 1972), Moore and Fung (1972) and Chiuderi and Riani (1974).

The present paper is an attempt to devise a theoretical model consistent with the observations reported in paper I (Boland et al., 1973) where a direct observation of non-thermal mechanical energy in the transition zone was reported and interpreted as being due to acoustic wave propagation although other explanations like turbulence could not be excluded. The resulting sound wave energy flux was in agreement with the estimated total energy loss from the corona and this gross energy balance affords the strongest evidence in favour of the hypothesis that mechanical waves transmitted from lower levels provide the energy which heats the corona. These observations have been confirmed by Boland et al. (1975) who also include MHD propagation in their analysis. The observations showed that the particle velocities associated with the mechanical energy are always subsonic. Hence the present paper does not assume the usual shock dissipation and the direct theoretical consequence of that hypothesis is examined in an attempt to determine the physical structure of the atmosphere which would result from a sound wave propagating beyond the upper chromosphere (temperature > 10^4 K). The calculation is based on an energy balance where the upper corona is heated by the damping of sound waves generated at the bottom of the atmosphere and cooling is by radiation from the lower part of the transition region to which the energy is transferred by thermal conduction. It is assumed that the atmosphere is in hydrostatic equilibrium and that it is spherically symmetric.

Finally it is shown that the upward pressure to which the atmosphere is subjected due to the momentum of the sound waves is greater than its base pressure. The solution to this dilemma may lie in (a) magnetic field effects (b) a more complete treatment of sound wave propagation (c) convective flow of material which will not greatly affect the overall energy balance.
2. Energy Balance

2.1. Energy Balance Equation

The total divergence of the energy flux is zero

\[
\frac{d}{dh} \left( \phi_s + \phi_C + \phi_R + \phi_w \right) = 0
\]  

where the subscripts refer to the sound wave, conduction, radiation and the solar wind. Since \( \phi_w \) is small compared to the total energy flux of the corona, it is neglected and the detailed energy balance throughout the atmosphere is formed between the other three components. Additionally, the total energy balance is formed between the input acoustic flux and the total radiated energy loss. Spherical symmetry is assumed and the geometry is included in the definition of

\[
\phi = \left[ \frac{R_\odot}{R_\odot + h_0} \right]^2 F,
\]  

where \( F \) is the normal energy flux (\( \text{erg cm}^{-2} \text{s}^{-1} \)) at height \( h \) (cm), \( R_\odot \) is the solar radius and \( h_0 \) is the height at which the model starts. \( \phi \) is therefore the energy flux (\( \text{erg s}^{-1} \)) contained in a solid angle, measured from the centre of the sun, that subtends 1 cm\(^2\) at the base of the model i.e. at radius \( R_\odot + h_0 \). The use of this energy flux parameter allows comparison with models derived by other authors.

2.2. Heat Conduction

The problem of heat conduction in the solar atmosphere has been discussed by Orrall and Zirker (1961). At the lowest temperatures considered (8000 K) the gas is not fully ionised. Nevertheless a coefficient of thermal conductivity \( \kappa \) is used which is valid for completely ionized hydrogen. This is considered to be reasonable since the calculations show that the conductive flux from the corona becomes small before departures from full ionisation become significant.

The heat flux equation is

\[
\phi_c = \kappa \left( \frac{R_\odot + h}{R_\odot + h_0} \right)^2 T^{5/2} \frac{dT}{dh},
\]  

where \( \kappa = 1.8 \times 10^{-5}/\ln \Lambda \) and \( \ln \Lambda \) is a quantity defined and tabulated by Spitzer (1956).

2.3. The Radiated Power

The problem of calculating the power radiated from a plasma has been treated by a number of authors including McWhirter and Hearn (1963). They solved the full ionisation-recombination equations for hydrogen-like ions and included the three forms of radiation viz. (a) Spectral lines arising from bound-bound transitions, (b) Recombination radiation arising from free-bound transitions, and (c) Bremsstrahlung arising from free-free transitions. The full time-dependent solution for other kinds of ions has not yet been done because of its complexity but simplified steady-state solutions have been published using stellar abundance (Pottasch, 1965; Cox and Tucker, 1969).

The radiated power from an optically thin plasma having a cosmic element abundance has been recalculated for the present paper. Particular attention was
paid to bound-bound transitions since these account for most of the power radiated at the temperatures considered. The adopted solar abundances (which include Fe) were based on those of Jordan (1966) and ionisation calculations were taken from Summers (1972) or Jordan (1969).

The total radiated power can be expressed by

$$\frac{dP_R}{dh} = n_e n(H) \left[ \frac{R_\odot + h}{R_\odot + h_0} \right]^2 P_{RAD},$$  \hspace{1cm} (2.4)

where $P_{RAD}$ is the radiated power function whose calculated value is plotted against temperature in Fig. 1, together with the results of Pottasch (1965) and Cox and Tucker (1969) for comparison.

It may be noted that two simple approximate expressions for $P_{RAD}$ can be derived from the calculations shown in Fig. 1. Over the temperature range $1.5 \times 10^6$ K to $\sim 10^7$ K it can be represented within a factor of 2 by

$$P_{RAD} \sim 1.8 \times 10^{-22} \text{ erg cm}^3 \text{ s}^{-1}. \hspace{1cm} (2.5)$$

Alternatively it may be represented to within a factor 3 over the temperature range from $1.5 \times 10^6$ K to over $10^7$ K by

$$P_{RAD} \sim 5 \times 10^{-20} T^{-4} \text{ erg cm}^3 \text{ s}^{-1}. \hspace{1cm} (2.6)$$

Hydrogen is fully ionised over most of the solar atmosphere considered. However, at temperatures less than $2 \times 10^4$ K, towards the base of the transition region, the population of atomic hydrogen becomes significant. In order to ensure a smooth transition into this lower region the fractional ionisation of hydrogen and helium has been taken into account. The values used are given in Table 1 and are based on atomic data by Bates et al. (1962).

Table 1. Values of the ionisation balance ratio

<table>
<thead>
<tr>
<th>$T$ (K)</th>
<th>$n(H)/n_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7.94 \times 10^5$</td>
<td>$4.80 \times 10^4$</td>
</tr>
<tr>
<td>$1.26 \times 10^6$</td>
<td>$1.85 \times 10^3$</td>
</tr>
<tr>
<td>$2.00 \times 10^6$</td>
<td>$1.09$</td>
</tr>
<tr>
<td>$3.16 \times 10^6$</td>
<td>$8.80 \times 10^{-1}$</td>
</tr>
<tr>
<td>$5.01 \times 10^6$</td>
<td>$8.70 \times 10^{-1}$</td>
</tr>
<tr>
<td>$7.94 \times 10^6$</td>
<td>$7.94 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
| $1.26 \times 10^7$ | $7.69 \times 10^{-1}$ |}

2.4. Sound-wave Dissipation

In evaluating the damping coefficients for sound waves the treatment by Lamb (1962) has been used. He derives a distance $l$ over which the sound wave flux falls by a fraction 1/e due to damping. Substituting the appropriate expressions for the thermal conductivity (Spitzer, 1967) and viscosity (Linhart, 1960) of fully ionised plasmas leads to a damping length

$$l = 96.3 \frac{n}{T} \text{ (Period)}^2 \text{ cm}.$$

The sound wave dissipation term in Eq. (2.1) is therefore (with a sound wave period of 300 s)

$$\frac{d\phi_s}{dh} = 1.16 \times 10^{-7} \frac{T}{n} \phi_s.$$  \hspace{1cm} (2.8)

It is of interest to note that the thermal conductivity term makes a much greater contribution (a factor 18) to dissipation than the viscosity term and it is appropriate to describe the damping as "thermal conductivity damping" rather than "viscous damping". This situation appears to be the case for all fully ionised plasmas whereas the reverse is true for gases. Another damping process is sound wave enhancement of radiation (McWhirter and Wilson, 1974) but this is small compared with the coefficient given in Eq. (2.8) and has been neglected.

3. Pressure Balance

3.1. Pressure Balance Equation

The pressure equilibrium of a static atmosphere carrying a sound wave under gravity is

$$\frac{d}{dh} (P_G + P_S) = \frac{G M_\odot}{(R_\odot + h)^2} \varrho,$$  \hspace{1cm} (3.1)

where

- $P_G$: the kinetic gas pressure,
- $P_S$: the sound wave pressure,
- $G$: the gravitational constant,
- $M_\odot$: the mass of the sun and
- $\varrho$: the local mass density.

3.2. Hydrostatic Equilibrium ($P_S = 0$)

As will be seen later, it did not prove possible to obtain a solution when the sound wave pressure term was included. Models are therefore presented for hydrostatic equilibrium (i.e. $P_S = 0$). Since $P_G = (n_e + n_i) kT$, integration of Eq. (3.1) from the base $h_0$ to a height $h$ gives

$$\int_{h_0}^{h} m_i \frac{G M_\odot}{1.56 n(H)(R_\odot + h)^{-2}} \, dh,$$

where $m_i$ is the mass of the hydrogen atom and the factors 1.15 and 1.56 take account of the presence of other elements besides hydrogen in the solar atmosphere (15% by number and 56% by mass). Although the atmosphere is fully ionized over most of the range considered, this is not so at the lower levels and, hence, the degree of ionization must be known for...
the application of Eq. (3.2). This problem has already been considered in Section 2.3 where the radiation losses were considered and values of \( n(H)/n_e \) are tabulated in Table 1.

4. Results

The problem is to solve the energy balance Eq. (2.1) and the equation of hydrostatic equilibrium (3.2). Those expressions provide one less equation than there are unknowns. Thus the solutions give rise to model atmospheres in which there is only one adjustable parameter, for example, \( \phi_{so} \) — the sound wave energy flux incident on the base of the atmosphere. Hence, the adoption of a value for this parameter will result in a uniquely defined model of both density and temperature with height i.e. each model is completely specified by its sound wave energy flux. Alternatively, a base pressure may be adopted which also results in a uniquely defined model including the required sound wave energy flux.

4.1. The Full Computer Solution

An analytical solution of the full set of equations was not possible and a computer was used to obtain a set of numerical solutions. An iterative method was developed because of the need to balance the total input of sound

---

**Table 2. Characteristics of the models A, B, C and D**

<table>
<thead>
<tr>
<th>Model</th>
<th>( \phi_{so} ) (erg s(^{-1}) (unit solid angle(^{-1}))</th>
<th>( n_e ) T (at 5 \times 10^6 cm) ( \times 10^{14} ) K</th>
<th>( n(H)_0 ) ( \times 10^3 ) cm(^{-3})</th>
<th>( T_{MAX} ) K</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.5 \times 10^5</td>
<td>3.74 \times 10^{14}</td>
<td>1.34 \times 10^{11}</td>
<td>2.27 \times 10^6</td>
</tr>
<tr>
<td>B</td>
<td>5 \times 10^5</td>
<td>6.36 \times 10^{14}</td>
<td>1.93 \times 10^{11}</td>
<td>2.61 \times 10^6</td>
</tr>
<tr>
<td>C</td>
<td>1 \times 10^6</td>
<td>1.11 \times 10^{15}</td>
<td>2.99 \times 10^{11}</td>
<td>2.98 \times 10^6</td>
</tr>
<tr>
<td>D</td>
<td>2 \times 10^6</td>
<td>1.97 \times 10^{15}</td>
<td>4.86 \times 10^{11}</td>
<td>3.35 \times 10^6</td>
</tr>
</tbody>
</table>

**Table 3. Details of the standard model**

<table>
<thead>
<tr>
<th>Height (km)</th>
<th>Temperature (K)</th>
<th>Electron density (cm(^{-3}))</th>
<th>Hydrostatic pressure (dynes cm(^{-2}))</th>
<th>Temperature gradient (K cm(^{-1}))</th>
<th>Conducted flux (erg s(^{-1}) (unit solid angle(^{-1}))</th>
<th>Mechanical flux (erg s(^{-1}) (unit solid angle(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.85 (3)</td>
<td>8.94 (3)</td>
<td>4.13 (6)</td>
<td>1.48 (12)</td>
<td>7.30 (5)</td>
<td>1.09 (0)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.88 (3)</td>
<td>1.34 (4)</td>
<td>4.04 (9)</td>
<td>1.32 (11)</td>
<td>3.36 (2)</td>
<td>1.07 (3)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.98 (3)</td>
<td>2.01 (4)</td>
<td>2.11 (10)</td>
<td>1.32 (11)</td>
<td>1.01 (1)</td>
<td>8.69 (3)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.88 (3)</td>
<td>3.02 (4)</td>
<td>1.56 (10)</td>
<td>6.22 (2)</td>
<td>3.23 (2)</td>
<td>1.37 (4)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.85 (3)</td>
<td>4.53 (4)</td>
<td>1.05 (10)</td>
<td>1.22 (1)</td>
<td>2.03 (2)</td>
<td>1.84 (4)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.89 (3)</td>
<td>6.80 (4)</td>
<td>1.36 (9)</td>
<td>1.30 (11)</td>
<td>1.25 (2)</td>
<td>3.03 (4)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.91 (3)</td>
<td>1.02 (5)</td>
<td>9.9 (9)</td>
<td>1.31 (11)</td>
<td>2.18 (2)</td>
<td>5.03 (4)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>1.97 (3)</td>
<td>1.53 (5)</td>
<td>3.24 (9)</td>
<td>1.29 (11)</td>
<td>6.92 (3)</td>
<td>7.21 (4)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>2.13 (3)</td>
<td>2.30 (5)</td>
<td>2.13 (9)</td>
<td>1.27 (11)</td>
<td>3.58 (3)</td>
<td>9.82 (4)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>2.60 (3)</td>
<td>3.45 (5)</td>
<td>1.37 (9)</td>
<td>1.12 (11)</td>
<td>1.81 (3)</td>
<td>1.31 (5)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>3.90 (3)</td>
<td>5.06 (5)</td>
<td>8.76 (8)</td>
<td>1.15 (11)</td>
<td>9.22 (4)</td>
<td>1.68 (5)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>5.86 (3)</td>
<td>6.47 (5)</td>
<td>6.38 (8)</td>
<td>1.07 (11)</td>
<td>5.89 (4)</td>
<td>1.94 (5)</td>
<td>3.00 (5)</td>
</tr>
<tr>
<td>8.79 (3)</td>
<td>7.89 (5)</td>
<td>4.82 (8)</td>
<td>9.88 (2)</td>
<td>4.04 (4)</td>
<td>2.16 (5)</td>
<td>2.99 (5)</td>
</tr>
<tr>
<td>1.32 (4)</td>
<td>9.37 (5)</td>
<td>3.65 (8)</td>
<td>8.91 (2)</td>
<td>2.88 (4)</td>
<td>2.35 (5)</td>
<td>2.98 (5)</td>
</tr>
<tr>
<td>1.98 (4)</td>
<td>1.10 (6)</td>
<td>2.74 (8)</td>
<td>7.82 (2)</td>
<td>2.06 (4)</td>
<td>2.50 (5)</td>
<td>2.96 (5)</td>
</tr>
<tr>
<td>2.97 (4)</td>
<td>1.27 (6)</td>
<td>2.01 (8)</td>
<td>6.64 (2)</td>
<td>1.48 (4)</td>
<td>2.60 (5)</td>
<td>2.92 (5)</td>
</tr>
<tr>
<td>4.45 (4)</td>
<td>1.45 (6)</td>
<td>1.43 (8)</td>
<td>5.40 (2)</td>
<td>1.05 (4)</td>
<td>2.64 (5)</td>
<td>2.84 (5)</td>
</tr>
<tr>
<td>6.68 (4)</td>
<td>1.64 (6)</td>
<td>9.75 (7)</td>
<td>4.17 (2)</td>
<td>7.28 (5)</td>
<td>2.58 (5)</td>
<td>2.69 (5)</td>
</tr>
<tr>
<td>1.00 (5)</td>
<td>1.84 (6)</td>
<td>6.32 (7)</td>
<td>3.02 (2)</td>
<td>4.67 (5)</td>
<td>2.38 (5)</td>
<td>2.43 (5)</td>
</tr>
<tr>
<td>1.51 (5)</td>
<td>2.02 (6)</td>
<td>3.89 (7)</td>
<td>2.04 (2)</td>
<td>2.81 (5)</td>
<td>2.00 (5)</td>
<td>2.01 (5)</td>
</tr>
<tr>
<td>2.26 (5)</td>
<td>2.17 (6)</td>
<td>2.26 (7)</td>
<td>1.28 (2)</td>
<td>1.48 (5)</td>
<td>1.43 (5)</td>
<td>1.44 (5)</td>
</tr>
<tr>
<td>3.39 (5)</td>
<td>2.28 (6)</td>
<td>1.26 (7)</td>
<td>7.45 (3)</td>
<td>5.81 (6)</td>
<td>8.15 (4)</td>
<td>8.19 (4)</td>
</tr>
<tr>
<td>5.08 (5)</td>
<td>2.34 (6)</td>
<td>6.73 (6)</td>
<td>4.09 (3)</td>
<td>3.49 (7)</td>
<td>3.27 (4)</td>
<td>3.30 (4)</td>
</tr>
<tr>
<td>7.63 (5)</td>
<td>2.35 (6)</td>
<td>3.56 (6)</td>
<td>2.18 (3)</td>
<td>7.81 (3)</td>
<td>8.11 (3)</td>
<td>8.11 (3)</td>
</tr>
<tr>
<td>1.14 (6)</td>
<td>2.36 (6)</td>
<td>1.92 (6)</td>
<td>1.18 (3)</td>
<td>9.76 (9)</td>
<td>7.08 (2)</td>
<td>9.67 (2)</td>
</tr>
<tr>
<td>1.72 (6)</td>
<td>2.36 (6)</td>
<td>1.10 (6)</td>
<td>6.75 (4)</td>
<td>2.42 (9)</td>
<td>2.42 (9)</td>
<td>*)</td>
</tr>
<tr>
<td>2.58 (6)</td>
<td>2.36 (6)</td>
<td>6.88 (5)</td>
<td>4.22 (4)</td>
<td>1.43 (9)</td>
<td>3.22 (1)</td>
<td>*)</td>
</tr>
<tr>
<td>3.87 (6)</td>
<td>2.36 (6)</td>
<td>4.74 (5)</td>
<td>2.95 (4)</td>
<td>6.18 (10)</td>
<td>2.37 (4)</td>
<td>*)</td>
</tr>
<tr>
<td>5.80 (6)</td>
<td>2.36 (6)</td>
<td>3.57 (5)</td>
<td>2.19 (4)</td>
<td>2.17 (10)</td>
<td>*)</td>
<td>*)</td>
</tr>
</tbody>
</table>

*) Negative values of conducted flux
wave energy with the total radiated energy. Numerical solutions were obtained for four values of the incident sound wave energy flux $\Phi_{ws}$ at the base of the atmosphere. The four solutions are designated models A, B, C and D and the corresponding $\Phi_{ws}$ values are given in Table 2 together with some particular characteristics of the models. The choice of the starting height $h_0$ and temperature $T_0$ is arbitrary and have been taken to be $h_0 = 2000 \text{ km}$ and $T_0 = 8000 \text{ K}$. This choice does not critically influence the results.

A fifth computer solution to be used as our standard model was obtained using the uppermost set of values of height, temperature and total pressure (viz. $1.85 \times 10^8 \text{ cm}$, $8930 \text{ K}$ and $0.148 \text{ dynes cm}^{-2}$) of the Harvard-Smithsonian Reference Atmosphere (Gingerich et al., 1971) as starting values for the present model. This turned out to require a sound wave energy flux of $3 \times 10^5 \text{ erg cm}^{-2} \text{ s}^{-1}$ and lies between models A and B. Details of the height distributions of pressure, density, temperature and energy flux are plotted in Fig. 2 (solid lines) and listed in Table 3. The curves show that whereas the values of temperature and density are varying rapidly with height in the transition region the pressure is relatively constant. Since density is important in determining the total radiated power loss the value of pressure is an important characteristic parameter of each model. Thus values of the product of electron density with temperature $n_e T$ at an arbitrarily chosen height of $5 \times 10^8 \text{ cm}$ are given in Table 2 together with the hydrogen density at the base of the atmosphere $n(H)_0$.

An interesting feature of the results is the comparative insensitivity of the temperature $T_{\text{MAX}}$ to the incident energy flux (see Table 2). Thus an eight-fold increase in $\Phi_{ws}$ gives rise to a 46% increase in the temperature. This is due to the rapid increase of thermal conductivity with temperature. This and a number of other features may best be understood by simplifying the problem to the point where an analytical solution is possible (see below).

It is instructive to remove the sound wave damping and assume that energy is deposited as heat at a great height—above $10^{12} \text{ cm}$. With this assumption the computer results are shown as dashed lines on Fig. 2. It may be noted that corresponding curves only differ significantly from each other at heights greater than
5.10^9 so that even at height of 6 \times 10^{11} \text{ cm (ten solar radii)} they depart by less than a factor 3. It may be concluded that the model is not sensitive to the spatial distribution of heat deposition provided it takes place at a sufficiently great height. In other words, the model is largely insensitive to the nature of the dissipation mechanism. Indeed, given the same basic assumptions, a similar model would result for any form of coronal heating which deposited its energy high in the atmosphere. This is because, over most of its range, the model is dominated by the balance between conductive dissipation and radiation.

### 4.2. Analytical Solution

An approximate analytical solution has been derived which displays the main features of the computer solution. This is possible because for most of the atmosphere the energy balance is dominated by the conduction and radiation terms. Further, in the important region of the transition zone, the pressure can be taken as constant. Making the further simplifying assumptions of plane parallel geometry and the full ionisation of hydrogen i.e. \( n(\text{H}) = n_e \) at all heights, the energy and pressure balance equations reduce to:

\[
\frac{d}{dh} \left( n_e T^{5/2} \frac{dT}{dh} \right) - \frac{A_R n_e^2 T^{-3/2}}{\kappa} = 0
\]

(4.1)

and

\[
n_e T = \text{constant} = \langle n_e T \rangle,
\]

(4.2)

where Eq. (2.6) is adopted for the radiation loss with the numerical coefficient represented by \( A_R \). The coefficients for radiation and thermal conductivity are retained so as to demonstrate scaling. The appropriate numerical values are \( A_R = 5 \times 10^{-29} \), \( \kappa = 1.0 \times 10^{-6} \) corresponding to \( \ln \Lambda = 18 \). The solution can be expressed in the following forms:

\[
T^2 \frac{dT}{dh} = \left( \frac{2 A_R}{\kappa} \right)^{1/4} \langle n_e T \rangle = 3.2 \times 10^{-7} \langle n_e T \rangle,
\]

(4.3)

\[
T^3 = \left( \frac{2 A_R}{\kappa} \right)^{1/4} \langle n_e T \rangle \ h = 9.5 \times 10^{-7} \langle n_e T \rangle \ h
\]

(4.4)

\[
\phi_e = \left( 2 A_R \kappa \right)^{1/4} \langle n_e T \rangle \ T^{1/4} = 3.2 \times 10^{-13} \langle n_e T \rangle^{1/4}.
\]

(4.5)

Comparison with the computer solutions for energy deposited at great height, shows that the analytical solution is a good representation of the model. Its value lies in being able to quickly see the functional form of the solution and being able to estimate important parameters easily.

Another instructive form of the analytical solution is obtained by combining (4.4) and (4.5) to give

\[
\phi_e = 3.2 \times 10^{-14} \langle n_e T \rangle^{7/16} h^{1/16}.
\]

(4.6)

If we consider this when \( \phi_e \sim \phi_{\nu o} \) i.e. at great heights, we see that the pressure \( \langle n_e T \rangle \) of the atmosphere is mainly determined by the input heating flux because of its insensitive dependence on height \( h \). The computer solutions show that the temperature maximum, and hence the height at which most of the energy has been deposited, occurs at \( h \sim 3 \times 10^{10} \text{ cm} \). However, since the expression for \( \phi_e \) increases very slowly with \( h \), the result is insensitive to the value adopted. Fig. 3 shows the relationship between the energy flux \( \phi_e \) and the "pressure" \( \langle n_e T \rangle \) together with the four points obtained from the computer solution and listed in Table 2. The numerical agreement (depending on the choice of \( h \)) may be fortuitous but the curve is seen to present the correct functional relationship (independent of \( h \)) between the energy flux and the pressure required to allow that energy to be radiated. A similar relation can be obtained by adopting a temperature instead of a height and using Eq. (4.5).

### 5. Comparison with Observations

#### 5.1. Electron Density

One of the results of the theoretical calculation given in Section 4 is the prediction of the electron number density as a function of height in the atmosphere. The values for models A, B, C and D are plotted in Fig. 4.

A number of measurements of the electron density as a function of height in the corona have been made at times of eclipse by observing the spatial variation of the polarization and intensity of the K-component—the
fraction of the photospheric visible radiation that is Thomson scattered by the electrons in the corona. These measurements have been inter-related and averaged by Allen (1973). His values are plotted as points on Fig. 4 and the agreement is satisfactory over a height range stretching from a hundredth of a solar radius to one solar radius above the limb. Compared with the predictions of our standard model the quiet equatorial values are within a factor three. Over a similar range of height Gabriel (1971) has recently calculated the density distribution of electrons from observations of the intensity of Lyman \( \alpha \) radiation suffering fluorescent scattering by the residual atomic hydrogen in the corona. His results are in good agreement with, and provide valuable support for the values given by Allen.

Also plotted on Fig. 4 are the electron density values of the Harvard Smithsonian Reference Atmosphere. These appear as the short curve at densities greater than \( 3 \times 10^{10} \) cm\(^{-3} \) and heights less than \( 2 \times 10^5 \) cm.

5.2. Temperature

The temperature distributions predicted by the theoretical models may be compared with those derived from an analysis of the observed intensities of spectral lines emitted from the transition region and corona. Two such analyses have been done by Jordan (1965) and by Dupree and Goldberg (1967) and their derived temperature distributions are plotted in Fig. 5 together with the present predictions for comparison. The chromospheric temperature corresponding to the Harvard Smithsonian Reference Atmosphere is also shown (heights < \( 2 \times 10^8 \) cm).

The temperature distributions in the theoretical models compare satisfactorily with those derived from the observed spectra only in gross terms. Although a sharp transition zone and maximum temperature of right order are found, there are considerable differences in detail, particularly at intermediate temperatures where the theoretical curves tend to lie below the others. Also, differences occur in the height at which the maximum temperature occurs, that derived by Jordan (\( \sim 5 \times 10^5 \) cm) being lower than that derived theoretically. The height predicted by the theory depends critically on the heating mechanism and for sound waves, on their period. If heating is at a comparatively low height the outer layers can only be heated by conduc-
tion outwards. Even the modest fluxes required to make up the losses by radiation and the solar wind over the very large distances would cause substantial thermal gradients and the outer layers would reach an unrealistically low temperature.

3.3. Energy Balance

An instructive way of presenting the models derived by Jordan (1965) and Dupree and Goldberg (1967) is via the energy balance Eq. (2.1), the individual terms of which can be computed from the equations of conductive flow (2.3) and radiative loss (2.4) using the temperatures and densities given by their models. This gives the sound wave flux $\phi_s$ which is required to maintain an energy balance throughout those models. The results are plotted against temperature in Fig. 6 together with the values derived by Boland et al. (1973, 1975) from the observed widths of spectral lines. Also plotted are the sound wave fluxes required by the present models A, B, C and D.

It can be seen that there is agreement in gross terms in that the total energy fluxes of $2 \times 10^6$ erg cm$^{-2}$ s$^{-1}$ (Jordan), $1 \times 10^6$ (Dupree and Goldberg) and $5 \times 10^5$ (Boland et al.) probably lie within the errors. However, there are important differences in detail. Both models show an increase of energy flux in the region of $10^5$ K with a maximum at about $4 \times 10^5$ K. This positive divergence of flux means that the Jordan and Dupree and Goldberg models are dissipating conducted flux in the region $5 \times 10^4 - 3 \times 10^5$ K at a rate in excess of what the medium can cope with in the form of radiation.

If it is assumed that the disparity does not arise from the method of analysing the observations to derive the models then it presents a major disability to the present theory. Either the expressions used to calculate the conducted flux and radiated power are wrong by more than a factor ten or a very large energy term has been left out of account. That a large term is missing from the energy balance Eq. (2.1) has been proposed by Kuperus and Athay (1967) who suggest that the large conducted heat flux may be converted into turbulent motion in the atmosphere. This has been studied further by Bessey and Kuperus (1970), who considered thermally driven motions in an isothermal gravitational atmosphere. The more relevant case of a temperature gradient is a much more difficult theoretical problem.

A related question is whether the medium is stable against high conductive flux and, indeed, there is one instability due to runaway electrons that places a firm limit on the conductive flux that can be maintained. This arises because an electric field results from the temperature gradient whose value can be derived from Spitzer (1967) and Spitzer and Härm (1955) to be

$$E = 7 \times 10^{-5} \frac{dT}{dh} \text{ V cm}^{-1}.$$  \hspace{1cm} (5.1)

Fig. 6. Curves showing the derived energy fluxes required for energy balance by the model atmospheres found by Jordan (1965) (marked CJ) and by Dupree and Goldberg (1967) (marked DG) and based on the observed spectral intensities. Also shown are the fluxes according to the present work for models A, B, C and D and the observed mechanical energy fluxes of Boland et al. (1973) (circles) and Boland et al. (1975) (squares).

Again from Spitzer, a criterion against runaway can be obtained as

$$6.6 \times 10^8 \frac{T E}{n_e \ln A} < 0.1.$$  \hspace{1cm} (5.2)

Combining (5.1) and (5.2) and taking $n_e T = 6 \times 10^{14}$ and $\ln A = 18$, we have

$$T^2 \frac{dT}{dh} < 2 \times 10^{10} \text{(K}^3 \text{ cm}^{-1}).$$  \hspace{1cm} (5.3)

The criteria is met by both the theoretical and observationally derived curves and we can conclude that there is no reason for believing that such an instability is present in the transition zone.

6. Pressure Due to Sound Waves

The above calculations were made for the case of hydrostatic equilibrium, the sound wave pressure being omitted from the pressure Eq. (3.1). This was because its inclusion resulted in too great a pressure for the atmosphere to balance. The calculations leading to this conclusion, and its possible consequences, are discussed in this section.

In a medium carrying a sound wave, we consider a fixed point in space where, at time $t$, the local density
and velocity are $\varphi$ and $v$ respectively. Then the local, instantaneous momentum density is $\varphi v$ and the momentum flux is $\varphi v^2$. The sound wave pressure is then the time average of the momentum flux, i.e.

$$P_s = \langle \varphi v^2 \rangle_{t}.$$  \hfill (6.1)

The kinetic energy density is, by definition, $\frac{1}{2} \langle \varphi v^2 \rangle_{t}$. Since the potential energy component is equal to the kinetic, then the total energy density is $\langle \varphi v^2 \rangle_{t}$, viz. the same as $P_s$. Expressing the energy density in terms of the energy flux and wave velocity, we derive the sound wave pressure in the convenient form

$$P_s = \varphi/c.$$ \hfill (6.2)

We have been unable to trace the evolution of finite amplitude plane sound waves as they propagate through the inhomogeneous gas of the atmosphere. This problem is particularly difficult because the properties of the gas are themselves dependent on both the overall atmospheric structure and the sound wave energy flux at each point in space. It must suffice to investigate the magnitude of the sound wave pressure arising from reflection and attenuation using the density and temperature distributions obtained when sound wave pressure is ignored.

To make the calculation it is necessary to recognise that the steep temperature gradient in the transition zone appears as a boundary at which reflection and a change of wave velocity can occur. If we identify with the subscripts 1 and 2 the regions of the incident and transmitted waves respectively then it can easily be shown that the total pressure exerted on the boundary is

$$\frac{\varphi_2}{c_1} \frac{1 + R}{1 - R},$$ \hfill (6.3)

where $R$ is the coefficient of reflectivity at the boundary. It will be noted that $\varphi_2$ is the energy flux reaching the corona and is therefore equivalent to $\varphi_1$ in the energy balance Eq. (2.1) and that $c_1$ is the velocity of sound appropriate for the chromosphere.

An expression for the reflectivity $R$ has been derived by Landau and Lifshitz (1959) where the boundary layer is very much thinner than the sound wavelength. Since, for the sound wave frequency assumed in the present paper ($\sim 1/300$ s$^{-1}$), the wavelength is more than 3000 km, their treatment should be valid for the present case. For normal incidence, it gives

$$R = \frac{n_2 \sqrt{T_2} - n_1 \sqrt{T_1}}{n_2 \sqrt{T_2} + n_1 \sqrt{T_1}},$$ \hfill (6.4)

where $n$ is the particle density and $T$ the temperature of regions 1 and 2.

Thus there is a condition viz.

$$n_1 \sqrt{T_1} = n_2 \sqrt{T_2},$$ \hfill (6.5)

Table 4. Sound wave pressure $\varphi_2/C_1$ and gas pressure $nkT$ for the four models A, B, C and D

<table>
<thead>
<tr>
<th>Model</th>
<th>$\varphi_2$</th>
<th>$\varphi_2/C_1$</th>
<th>$nkT$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$2.5 \times 10^5$</td>
<td>$2.4 \times 10^{-1}$</td>
<td>$1.1 \times 10^{-1}$</td>
</tr>
<tr>
<td>B</td>
<td>$5 \times 10^5$</td>
<td>$4.9 \times 10^{-1}$</td>
<td>$1.8 \times 10^{-1}$</td>
</tr>
<tr>
<td>C</td>
<td>$1 \times 10^6$</td>
<td>$9.7 \times 10^{-1}$</td>
<td>$3.2 \times 10^{-1}$</td>
</tr>
<tr>
<td>D</td>
<td>$2 \times 10^6$</td>
<td>1.9</td>
<td>$5.5 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

where $R = 0$ and all the energy is transmitted. For this condition to be met the density in the corona must be greater than that corresponding to the constant pressure model. Even with $R = 0$ a pressure $\varphi_2/c_1$ is exerted on the atmosphere due to the absorption of the incident sound wave momentum. Values of $\varphi_2/c_1$ are listed in Table 4 for each of the four models A, B, C and D together with values of the total plasma pressure ($nkT$) in the transition region ($T \sim 3 \times 10^6$ K).

The sound wave pressure is in each case between two and four times the plasma pressure which, in the hydrostatic equilibrium models, supports the weight of the whole corona. An attempt was made to include the sound wave term in the pressure balance equations but resulted in such high densities that the increased radiated power loss could not be balanced and no equilibrium solution was possible. The inclusion of the effect of the reflected component makes a solution even more difficult. It should be noted that the sound wave pressure tends to increase the density above the interface and hence to tend to satisfy the condition (6.4) that there should be no reflection.

There are three possible solutions to this dilemma:

(a) A full solution of the sound wave equations in an atmosphere having steeply varying parameters may result in a much smaller force being exerted.

(b) The coronal heating is by magnetic or magneto-acoustic waves which, because they have a higher propagation velocity, would exert a correspondingly smaller pressure.

(c) The conducted flux is returned to the transition region only over a small fraction of the total surface by the effect of channelling by the magnetic field associated with the chromospheric network [following a similar suggestion by Kopp (1972)].

The first possibility (a) would require a detailed analysis of the problem and is outside the scope of the present paper. The other two depend on magnetic fields and are discussed in the next section.

7. Magnetic Field Effects

So far, the sun's magnetic field has been ignored but its possible effect will be considered now. The coefficient of thermal conductivity is modified because of the large ratio of conductivity along the field lines to that across the field. The ratio may approach unity at the bottom
of the transition region but elsewhere is very much larger. Thus the conducted flux follows the field and is channelled along it from the corona into areas of high field in the intergranular regions of the chromospheric network (Kopp and Kuperus, 1968). A detailed calculation taking account of these ideas depends on the magnetic field configuration which is irregular and not well enough known. However, it is possible to draw some general conclusions.

Thermal conductivity along the field lines is numerically close to that in a magnetic-field-free region so that the expression (2.3) may be used. Because the magnetic field concentrates the heat flux into the channels of high field only a fraction of the total solar surface is available for the conducted flux. This fraction varies with height; being smallest near the chromosphere but approaching unity in the corona. It is a consequence of this modification to the model that the same high coronal temperature can be maintained against a smaller heat flux averaged over the whole surface. It also means that the gradient is steeper at lower heights and less steep at greater heights as compared with the results presented in Section 4. The radiated power will come from regions of high heat flux and therefore of high magnetic field. In this respect the model is consistent with the observation that the chromospheric network is bright in its spectral lines. It follows that the total power radiated is less than it would be if the whole surface was involved and hence there is need for less energy to be deposited in the corona. As a consequence the average pressure on the transition zone due to the waves is less than it would be if radiation came from the whole surface. By assuming an arbitrary shape for the magnetic field and assuming channeling of heat flux it was found possible to obtain a computer solution including the term for sound wave pressure. It is felt however that the arbitrary nature of the assumptions makes this particular solution of doubtful value and it is not presented for this reason. The magnetic field makes possible magneto-hydrodynamic modes of energy propagation to heat the corona. Consideration of these in detail is outside the scope of this paper and in any case would be uncertain because of lack of information about the magnetic field. However it is clear that modes of propagation with much higher propagation velocities than that of sound are possible which would therefore reduce the momentum flux for a given energy flux. Thus the pressure due to these waves would be less than for sound waves.

8. Conclusion

An attempt has been made to produce an internally consistent theoretical model for the solar corona in which all the consequences of the physical assumptions are followed through. This objective has had to be compromised because of the impossibility of balancing the pressure equation when account is taken of the term due to the acoustic momentum flux. The difficulty has been avoided by omitting this term and basing the solution on hydrostatic equilibrium. This step is justified on the grounds that the solar magnetic field may modify the pure acoustic mode of propagation and thereby reduce the average momentum flux. Thus, a solution of the energy and pressure balance equations has been obtained. The equations are solved not only at every point throughout the atmosphere but also for the atmosphere as a whole so that the total mechanical energy flux that is absorbed is balanced by the total energy radiated by the atmosphere. The model is calculated using the upper point of the Harvard Smithsonian Reference Atmosphere as its base values. With these values and no other arbitrary assumptions, the model that is derived is successful in reproducing the main features of the temperature and density distributions estimated on the basis of observed data. Another result of the calculation shows that this specific model requires a mechanical energy input flux from the surface of the sun of $3 \times 10^5$ erg cm$^{-2}$ s$^{-1}$. This value is consistent with the observations of Boland et al. (1973, 1975) based on measurements of the Doppler broadening of lines arising in the transition zone.

Acknowledgements. We are grateful to colleagues at the Appleton and Culham Laboratories for discussions about this work and to the computer staffs at the Appleton and Atlas Laboratories for their cooperation. The Appleton Laboratory contribution to this work is published with the permission of the Director of that Laboratory.

References

Athay, R. G. 1969, Solar Phys. 9, 51
Bessey, R. J., Kuperus, M. 1970, Solar Phys. 12, 216
Biermann, L. 1948, Zeit, F. Astrophys. 25, 161
Chiuderi, C., Riani, I. 1974, Solar Phys. 34, 113
Cox, D. P., Tucker, W. H. 1969, Space J. 157, 1157
Gabriel, A. H. 1971, Solar Phys. 21, 392
Gingerich, O., Noyes, R. W., Kalkofen, W. 1971, Solar Phys. 18, 347
Jordan, C. 1965, Ph. D. Thesis, Univ. of London (see also Jordan and Wilson, 1971)
Kopp, R. A. 1968, Report AFCLR 68-0312
Kopp, R. A. 1972, Solar Phys. 27, 373
Kuperus, M. 1969, Space Sci. Rev. 9, 713
A Model for the Solar Corona Based on Energy Balance

Schatzmann, E. 1949, Ann. Astrophys. 12, 203
Schwarzschild, M. 1948, Astrophys. J. 107, 1
Spitzer, L., Härm, R. 1953, Phys. Rev. 89, 977
Ulmschneider, P. 1970, Solar Phys. 12, 403
Ulmschneider, P. 1971, Astron. & Astrophys. 12, 297

R. W. P. McWhirter
Appleton Laboratory
Astrophysics Research Division
Abingdon, Oxfordshire OX 14 3 DB
United Kingdom

P. C. Thonemann
Department of Physics
University College of Swansea
Singleton Park, Swansea
United Kingdom

R. Wilson
Department of Physics and Astronomy
University College London
Gower Street
London WC 1 E 6 BT
United Kingdom

© European Southern Observatory • Provided by the NASA Astrophysics Data System
Numerical Solution of Radiative Transfer Equation in Extended Spherical Atmospheres with Rayleigh Phase Function

A. Peraiah
Lehrstuhl für Theoretische Astrophysik der Universität, Heidelberg

Received January 15, revised March 7, 1975

**Summary.** A numerical solution of radiative transfer equation has been obtained in spherically symmetric homogeneous medium with Rayleigh's phase function in the frame work of discrete space theory of Grant and Hunt (1968) and Peraiah and Grant (1973). The fast doubling algorithm has been used in spherical cases for large optical thicknesses and highly extended spherical shells. Fluxes have been conserved to the machine accuracy except in the case of large values of \( B/A \) (ratio of outer to inner radius of the atmosphere) where doubling algorithm has been used and flux is conserved up to few units in the sixth decimal place. The linear polarization is found to be about 15% for \( B/A = 5 \) and \( \tau = 5-10 \) (where \( \tau \) is the total optical depth of the medium).

**Key words:** radiative transfer-spherical atmospheres-Rayleigh phase function

1. **Introduction**

Chandrasekhar (1950) gave an explicit mathematical formulation for the equation of transfer of linearly polarized radiation in plane parallel atmospheres. Further work on this problem was done by Code (1950), Harrington (1969, 1970) and others all of whom assumed plane parallel approximation.

The quasi-stellar object 3C446 showed a high degree of polarization (Burbidge and Burbidge, 1967, p. 77) of about 10%. In estimating fluxes from a “standard” QSO, they found (p. 121) that the radius of the shell to be about \( 10^{19} \) cm with a thickness of \( 10^{17} \) cm which suggests the presence of extended atmospheres attached to these objects (although the atmosphere is not highly extended, the presence of scattering complicates the calculation of the solution of radiative transfer equation). Recently Serkowski (1970) found polarization in the stars with extended atmospheres or spherical shells. Polarization has also been found in late type stars (Vardya, 1970 and see references given there). Polarization is caused by scattering by electrons in early type stars and by molecules in late type stars and both types of scatterings have the same angular distribution given by the solution of transfer equation with Rayleigh’s phase function. Consequently one must take scattering into account in calculating the transfer of linearly polarized radiation. Whenever there is scattering in transferring radiation, we may have to iterate for the solution of the transfer equation. In problems dealing with spherical symmetry, the ray continuously changes its direction with the radius vector which amounts again to some sort of scattering, which we shall call curvature-scattering. This, taken together with the scattering either by electrons or by molecules would greatly complicate the process of emergence of radiation from such atmospheres. There have been some attempts towards these problems in the recent past (Cassinelli and Hummer, 1971; Schmidt-Burgk, 1973, and others). Peraiah and Grant (1973, henceforth called paper I) developed a method to calculate a direct numerical solution of radiative transfer equation in spherical shells based on the discrete space theory of Grant and Hunt (1969a,b). A slightly different version of this approach was used by Plass et al. (1973) and was found to be an extremely fast algorithm by which one can calculate the radiation field at any given point inside the medium directly.

We shall calculate the angular distribution of linear polarization in extended homogeneous spherical medium with Rayleigh’s phase function. The primary aim of this paper is just to see how curvature affects the emergent radiation field from a completely scattering medium. However, one cannot compare these results with those of observations as one observes fluxes from either a distorted surface of a star or from one of the components of a close binary system during eclipse where as we calculate mainly the angular distribution of the specific intensities. The polarization of distorted stars with extended atmospheres will be treated in a forthcoming paper.

© European Southern Observatory • Provided by the NASA Astrophysics Data System
2. Discretization of Radiative Transfer Equation

The transfer equation in divergence form in spherical symmetry is

\[
\frac{\mu}{r^2} \frac{\partial}{\partial r} \left\{ r^2 I(r, \mu) \right\} + \frac{1}{r} \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) I(r, \mu) \right\} + \sigma(r) I(r, \mu) = \sigma(r) \left[ 1 - \sigma(r) \right] b(r) + \frac{\sigma(r)}{2} \int_{-1}^{1} P(r, \mu', \mu) d\mu' ,
\]

where \( \sigma(r) \) is the albedo for single scattering, \( I(r, \mu) \) is the specific intensity, \( r \) = radius, \( \mu = \cos \theta \), \( \sigma(r) \) is the absorption coefficient, \( b(r) \) is the source inside the medium and \( P(r, \mu, \mu') \) is the phase function.

If we write

\[
U(r, \mu) = r^2 I(r, \mu)
\]

and set \( \sigma(r) = 1 \), (as we are considering only scattering) we can rewrite Eq. (1) as,

\[
\mu \frac{\partial U(r, \mu)}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) U(r, \mu) \right\} + \sigma(r) U(r, \mu) = \frac{\sigma(r)}{2} \int_{-1}^{1} P(r, \mu', \mu) U(r, \mu') d\mu'
\]

for positive \( \mu \in (0, 1) \).

And

\[
-\mu \frac{\partial U(r, -\mu)}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) U(r, -\mu) \right\} + \sigma(r) U(r, -\mu) = \frac{\sigma(r)}{2} \int_{-1}^{1} P(r, -\mu, \mu) U(r, \mu') d\mu'
\]

for the oppositely directed beam.

If the radiation field is represented by two perpendicularly polarized intensity beams, then

\[
U(r, \mu) = \begin{bmatrix} U_L(r, \mu) \\ U_R(r, \mu) \end{bmatrix} ,
\]

where \( U_L \) and \( U_R \) refer respectively to the states of polarization in which the electric vector vibrates along and perpendicular to the principle meridian. The phase function is given by (see Grant and Hunt, 1968),

\[
P(r, \mu, \mu') = \frac{3}{2} \left( 2(1 - \mu^2)(1 - \mu'^2) \right) \mu'^2
\]

\[
= \begin{bmatrix} P_{11}(\mu, \mu') & P_{12}(\mu, \mu') \\ P_{21}(\mu, \mu') & P_{22}(\mu, \mu') \end{bmatrix} .
\]

Now the transfer equation for each component \( U_L \) and \( U_R \) can be written as,

\[
-\mu \frac{\partial U_L(r, \mu)}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) U_L(r, \mu) \right\} + \sigma(r) U_L(r, \mu) = \frac{\sigma(r)}{2} \int_{-1}^{1} P_{11}(\mu, \mu') U_L(r, \mu') + P_{12}(\mu, \mu') U_R(r, \mu') d\mu'.
\]

and

\[
-\mu \frac{\partial U_R(r, -\mu)}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \mu} \left\{ (1 - \mu^2) U_R(r, -\mu) \right\} + \sigma(r) U_R(r, -\mu) = \frac{\sigma(r)}{2} \int_{-1}^{1} P_{21}(\mu, -\mu') U_L(r, \mu') + P_{22}(\mu, -\mu') U_R(r, \mu') d\mu'.
\]

The equations for \( U_R \) are similar.

The discrete representation of Eqs. (7) and (8) together with the two similar equations for \( U_R \) is written following paper I as,

\[
M^* \left[ U_{n+1}^+ - U_n^+ \right] + \epsilon_1 \left[ \sum_{n}^+ \sum_{n}^+ A_{n+1}^+ U_{n+1}^- + \sum_{n}^+ \sum_{n}^+ A_n^- U_{n}^+ \right] + \tau_{n+\frac{1}{2}} U_{n+1}^+ = \frac{1}{2} \tau_{n+\frac{1}{2}} \left[ P_{n+1}^+ c^* U_{n+1}^+ + P_{n+1}^- c^* U_{n+1}^- \right] \]

and

\[
M^* \left[ U_{n-1}^- - U_n^- \right] + \epsilon_1 \left[ \sum_{n}^+ \sum_{n}^+ A_{n}^+ U_{n}^- + \sum_{n}^+ \sum_{n}^+ A_n^+ U_{n}^- \right] + \tau_{n+\frac{1}{2}} U_{n-1}^- = \frac{1}{2} \tau_{n+\frac{1}{2}} \left[ P_{n-\frac{1}{2}}^+ c^* U_{n-\frac{1}{2}}^+ + P_{n-\frac{1}{2}}^- c^* U_{n-\frac{1}{2}}^- \right] ,
\]

where

\[
U_n^\pm = \begin{bmatrix} U_n^+(L) \\ U_n^+(R) \end{bmatrix} ,
\]

\[
U_n^+(L) = \begin{bmatrix} U_{n+1}^-(L) \\ U_{n+2}^-(L) \\ \vdots \\ U_{n+m}^-(L) \end{bmatrix} \quad \text{and} \quad U_n^-(L) = \begin{bmatrix} U_{n-1}^+(L) \\ U_{n-2}^+(L) \\ \vdots \\ U_{n-m}^+(L) \end{bmatrix} ,
\]

\[
U_{n,m}(L) = U_L(r_n, \pm \mu), \quad j = 1, 2, \ldots, m .
\]

The vector \( U_n^+(R) \) is similarly defined. And,

\[
M^* = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} , \quad M = \begin{bmatrix} \mu_{j} \delta_{jk} \end{bmatrix} ; \quad c^* = \begin{bmatrix} c \end{bmatrix} ,
\]

\[
c = \begin{bmatrix} c_1 \delta_{jk} \end{bmatrix} ,
\]

\[
j, k = 1, 2, \ldots, m .
\]

where \( \mu_{j} \) and \( c_1 \) are the roots and weights of a suitable quadrature formula. Furthermore,

\[
P_{n+\frac{1}{2}}^+ = P_{n+\frac{1}{2}}(-\mu_j, \mu_k) = P_{n+\frac{1}{2}}^- |_{n+\frac{1}{2}}
\]

\[
P_{n+\frac{1}{2}}^- = P_{n+\frac{1}{2}}(\mu_j, -\mu_k) = P_{n+\frac{1}{2}}^- |_{n+\frac{1}{2}}
\]

and

\[
P_{n+\frac{1}{2}}^+ = \begin{bmatrix} P_{11}^+ & P_{12}^+ \\ P_{21}^+ & P_{22}^+ \end{bmatrix}
\]

with similar expressions for \( P_{n+\frac{1}{2}}^+, P_{n+\frac{1}{2}}^-, P_{n+\frac{1}{2}}^+ \), \( \alpha, \beta = 1, 2; \ \mu_j, \mu_k > 0 \),

\[
\tau_{n+\frac{1}{2}} = \int_{r_{n+\frac{1}{2}}}^{r_{n+\frac{1}{2}}} \sigma(r) dr = \sigma_{n+\frac{1}{2}}(r_{n+\frac{1}{2}} - r_{n+\frac{1}{2}}) .
\]
The curvature factor \( q_c \) is defined as
\[
q_c = \Delta r / \bar{r},
\]
where \( \Delta r \) is the thickness of the cell (geometrical) and \( \bar{r} \) is the mean radius of the cell. In all these calculations we have taken \( \bar{r} \) as the outer radius of the basic cell. And,
\[
A_\pm^* = \begin{bmatrix}
A_x \\
A_y
\end{bmatrix},
\]
where \( A_\pm \) are curvature matrices (see paper I or Periaiah, 1973). The average intensities over the cell are expressed as a weighted mean of the interface intensities. Thus,
\[
(I - X_{n+\frac{3}{2}}^+ U_{n+\frac{1}{2}}^+ U_{n+\frac{1}{2}}^-) = U_{n+\frac{1}{2}}^+,
\]
\[
(I - X_{n+\frac{3}{2}}^- U_{n+\frac{1}{2}}^- U_{n+\frac{1}{2}}^+) = U_{n+\frac{1}{2}}^-,
\]
where \( X_{n+\frac{3}{2}}^\pm \) are \( 2m \times 2m \) diagonal matrices with the structure
\[
X_{n+\frac{3}{2}}^\pm = \begin{bmatrix}
X_{n+\frac{3}{2}}^\pm(L) & \\
X_{n+\frac{3}{2}}^\pm(R)
\end{bmatrix}
\]
where \( X_{n+\frac{3}{2}}^\pm \) are diagonal \( m \times m \) matrices. Usually, we choose \( X_{n+\frac{3}{2}}^+ = \frac{1}{2} I \) (I is the identity matrix) for diamond scheme and \( X_{n+\frac{3}{2}}^- = I \) for “step” scheme (Carlson, 1963). Other choices of intermediate character are available (Grant, 1968).

It is now straightforward to calculate the transmission and reflection matrices for a given shell of thickness \( \tau \) from Eqs. (9) and (10) following paper I [see Eqs. (2.13) and (2.14) and the Appendix].

Flux conservation is ensured by satisfying the necessary condition [see the last of Eq. (3.6) of paper I] that
\[
\| S(n, n + 1) \| = 1
\]
which, in terms of transmission and reflection matrices (see Appendix of paper I), becomes
\[
\| T(n + 1, n) + R(n + 1, n) \| = 1
\]
\[
\| T(n, n + 1) + R(n, n + 1) \| = 1.
\]

And this leads to the two relations [Eq. (4.3) of paper I]
\[
(1) \sum_{j=1}^{m} c_j (A_{jk}^- - A_{jk}^+) = 0 \quad \text{for all } k
\]
and [from Eq. (3.9) of Grant and Hunt, 1969b]
\[
(2) \sum_{j=1}^{m} c_j [P_{11}(\mu_j, \mu_k) + P_{21}(\mu_j, \mu_k)]
+ P_{11}(-\mu_k, \mu_k) + P_{21}(-\mu_j, \mu_k) = 1 \quad \text{for all } k,
\]
and
\[
\sum_{j=1}^{m} c_j [P_{21}(\mu_j, \mu_k) + P_{22}(\mu_j, \mu_k) + P_{12}(-\mu_j, \mu_k)]
+ P_{22}(-\mu_j, \mu_k) = 1 \quad \text{for all } k,
\]
and two other similar equations for \( P_{n+\frac{3}{2}}^- \) and \( P_{n+\frac{3}{2}}^+ \).

The right hand side of Eq. (17) need not be exactly 1 because the calculation of the \( p \) matrices depend upon \( \mu_j^p \) which are taken from any quadrature formula depending upon the requirement of the problem. If this is not satisfied exactly, renormalization of the elements of these matrices is necessary (see Plass et al., 1973). We have used the zeroes and weights of Gauss-Legendre quadrature on \([0, 1]\) and the relations (17) are satisfied to the machine accuracy.

The emergent radiation field can be calculated either by the internal field algorithm or by the external field algorithm (Grant and Hunt, 1968). The former calculates the radiation field at any point inside the medium at the expense of large machine storage space and the latter calculates only the emergent radiation without any need of storage space, to the same accuracy. We have used both algorithms depending upon the result that is sought.

3. Results and Discussion

Calculations have been made for optical depths up to 100 and the ratio \( B/A \) (ratio of outer to inner radius of the spherical medium) has been taken from 1 to 5. Fifty discrete points along the radial direction are chosen \((N = 50)\). The step size \( \Delta \tau \) is chosen so that
\[
\Delta \tau < \Delta \tau_{\text{crit}} = \min_j \mu_j \left( \frac{1}{(1 - \omega_{p,j})^3} \right)
\]
to ensure non-negativity and hence stability of the algorithm. The curvature factor of the outermost shell is
\[
q_{\text{out}} = \frac{B - A}{NB}
\]
and in terms of \( q_{\text{out}} \) we can calculate \( q_n \) for any shell \( n \) as
\[
q_n = \frac{q_{\text{out}}}{1 - (n - 1)q_{\text{out}}}.
\]

To maintain stability and non-negativity of the solution, we must obtain non-negative transmission and reflection matrices (for simplicity we write \( t \) and \( r \) matrices) which satisfy the relation (15) [see Section (3) of paper I and their calculation involves the stepsize restriction given by the inequality (18) containing the curvature factor \( q \) as defined in (19) and (20). This forces us to select small \( q \)’s in consequence of which we have to employ a large number of shells. Therefore we have to use a proportionately large machine storage space as the diffuse transmission and reflection matrices are to be stored at the boundary of each shell.

We divide the medium of interest into \( N \) shells (where \( N \) depends mostly on the machine capacity) and if in any shell, \( \Delta \tau > \Delta \tau_{\text{crit}} \) then subdivide it into smaller shells until, in each shell the condition \( \Delta \tau < \Delta \tau_{\text{crit}} \) is satisfied. Now, the \( t \) & \( r \) matrices for each subshell with its \( q \) are
calculated and they are added by star algorithm [see Eqs. (3.7) and (3.8), paper 1] to obtain the $t$ & $r$ matrices for the original shell. However, from Eqs. (19) and (20), we see that $q$ increases from the surface ($n = 1$) to the bottom ($n = N$) of the atmosphere and hence the number of subshells in each shell increases and so does the number of star additions—a time consuming process. So, instead of calculating the $t$ & $r$ matrices for each of the subshells with different $q$'s, we can calculate them by using an average $q$ of all subshells and add them by star product, doubling the shells everytime we use the star algorithm which we shall call doubling process. By this process, we can save about 50% on the computing time but with some loss of accuracy. The flux could be conserved up to few units in the sixth decimal place. The system seems to be quite stable even when we use quite large optical thicknesses and highly extended spherical shells.

After we calculate the $t$ & $r$ matrices, the radiation field can be computed by means of either internal field or external field algorithms with the boundary conditions given below.

No incident radiation has been given at $T = 0 (n = 1)$.

\[ U_1^+(L) = 0 \]
\[ U_1^-(R) = 0 \quad \text{for all } \mu_j \]

and an unpolarized radiation is incident at $\tau = T (n = N + 1)$.

\[ U_{N+1}^+(L) = 1 \]
\[ U_{N+1}^-(R) = 1 \quad \text{for all } \mu_j \]

so that the flux

\[ \sum_{j=1}^{m} U_{N+1}^+(L \text{ or } R) \mu_j c_j = 1. \]

---

**Fig. 1.** The angular distribution of specific intensities $(I_L = U_L/r^3$ and $I_R = U_R/\rho^3)$ for the radial coordinate $r_c$ ($n = 1$ to 50) for $B/A = 1$ and $\tau = 10$. Dashed curves represent $I_L$ and continuous curves represent $I_R$. Resolution between $I_L$ and $I_R$ can be found only from $n \leq 5$. Numbers represent $n$.

**Fig. 2.** Same as in Fig. 1 except that $B/A = 5$, i.e., spherical case. Notice that $I_L$ and $I_R$ are resolved all along the radius except deep inside the medium. Notice also that at $\mu = \pm 1$ or $-1$, $I_L = I_R$ which is true for $B/A = 1$ in Fig. 1 also. Numbers refer to $n$.

**Fig. 3.** Angular distribution of the emergent radiation for the indicated values of $B/A$. Dashed curves represent $I_L$ and continuous curves represent $I_R$. Notice that at $\mu = 1$, $I_L = I_R$. 

© European Southern Observatory • Provided by the NASA Astrophysics Data System
The degree of polarization $P_n$ of radiation from any shell $n$ is calculated by the relation

$$P_n^\pm = \frac{U_n^\pm (R) - U_n^\pm (L)}{U_n^\pm (R) + U_n^\pm (L)}.$$  \hspace{1cm} (21)

The results are presented in Figs. 1–6. In Figs. 1 and 2 we have shown the angular distribution of specific intensities $I_R$ and $I_L ((U_R, U_L)/r^2)$ along the radial direction for plane parallel and spherical cases respectively, for the indicated parameters of $\tau$ and $B/A$. One can notice that the differences between $I_R$ and $I_L$ are larger in spherical system than in plane parallel systems and hence larger polarization in the former systems. This is due to the fact that curvature scattering enhances the effects of Rayleigh’s scattering. In Fig. 3 we have given the angular distribution of the emergent radiation for $B/A = 1, 3, 5$ and $\tau = 5$ which is quite similar in its nature of variation to that in Fig. 4 of paper I. We notice that the difference between $I_L$ and $I_R$ diminishes as we go from $\mu \approx 0$ to $\mu \approx 1$, that is, the polarization increases towards the limb. In Fig. 4 the radial distribution of polarization defined by Eq. (21) is plotted against $\mu \in [-1, +1]$. Here $n = 50$ and $n = 1$ correspond respectively to the bottom and surface of the atmosphere. One can notice that both in plane parallel and spherical cases there is a progressive increase in polarization towards the surface. In plane parallel case the polarization is less than that in spherical case at a given point in the atmosphere. However, the maximum of polarization occurs around $\mu \approx 0$ more towards into the atmosphere. This is more so in plane parallel than in spherical case. So, one should generally be able to observe polarization in stars with extended atmospheres which explains the fact that Serkowski (1970) found polarization only in the stars with extended shells or atmospheres. The emergent angular distribution of polarization given in Eq. (21) is plotted for $B/A = 1, 3$ and 5 in Fig. 5. There is a substantial difference between polarization for $B/A = 1$ and that for $B/A = 3$ and a further increase in $B/A$ from 3 to 5 does not significantly increase the polarization as fast as it was from that at $B/A = 1$ to.
that at $B/A = 3$. Figure 6 gives the trend of polarization with respect to $B/A$ for each ray. Again, in spherical case we find more polarization than in plane parallel case. From these results we notice that the polarization could be as large as 15%. One must however investigate the polarization of radiation from the distorted surface of a star which is under consideration.

Acknowledgments. I wish to thank Dr. I. P. Grant of Pembroke College, Oxford, England, Professor G. Traving, Professor B. Babad, Dr. J. Schmidt-Burgk and W. H. Kegel for helpful comments and criticism on the manuscript.

This work has been performed as part of the Sonderforschungsbereich 132 “Theoretische und praktische Stellarspektroskopie” which is sponsored by the Deutsche Forschungsgemeinschaft.

References

Burbidge, G., Burbidge, M. 1967, Quasi-Stellar Objects, W.H. Freeman and Company; San Francisco and London

Chandrasekhar, S. 1950, Radiative Transfer, Oxford University Press
Grant, I.P. 1968, J. Comp. Phys. 2, 251
Plass, G.N., Kattawar, G.W., Catchings, F.E. 1973, Appl. Optics. 12, 314

A. Peraiah
Lehrstuhl für Theoretische Astrophysik
der Universität Heidelberg
D-6900 Heidelberg 1
Im Neuenheimer Feld 294
Federal Republic of Germany