VISCOSITY AND THE CHEW-GOLDBERGER-LOW EQUATIONS IN THE SOLAR CORONA

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ABSTRACT

We provide a general discussion of the dominant terms in the stress tensor in a magnetized plasma such as the solar corona. The importance of dissipative terms such as electrical resistivity, heat conduction, and inter-species collisions is assessed. For average coronal conditions, the proton stress tensor is found to reduce to the dominant terms in the classical expression for the viscous stress. The classical expression can fail in the transition region, however. In the diffusion region of reconnection, classical viscosity will be appropriate if the resistivity is very large, so that the diffusion region is broad, but in that case the viscous heating is small compared to the resistive heating. On the other hand, the more general expression for the stress tensor is required if the diffusion region is thin; the stress tensor will be important in this case. We also consider the electron stress tensor and show how the classical expression for electron viscosity can fail in the transition region and lower corona.

Subject headings: plasmas — Sun: corona

1. INTRODUCTION

Studies of solar coronal dynamics usually assume scalar pressure and ignore viscosity. There are, however, conditions under which coronal viscosity can be significant, and its omission may lead to misleading results.

In the corona, viscosity is due mainly to the protons. The inequality \( \omega_{cp} \tau_p \gg 1 \) holds, where \( \omega_{cp} \) is the proton cyclotron frequency and \( \tau_p \) is the mean time between momentum-changing collisions. From Braginskii (1965)

\[
\tau_p = 0.75 T_p^{3/2}/n s
\]

for an electron-proton plasma; \( T_p \) is the proton temperature, \( n \) is the proton (or electron) concentration (all units are cgs), and the Coulomb logarithm has been taken to be 22. For example, in an active region loop we might have \( T_p = 2.5 \times 10^6 \) K, \( n = 3 \times 10^9 \) cm\(^{-3}\), and the magnetic field strength might be \( B = 50 \) G. Then \( \omega_{cp} \tau_p = 4.7 \times 10^7 \). If \( \omega_{cp} \tau_p \gg 1 \), then the viscous stress tensor is

\[
S_{ij} = 3 \eta_0 \left( \frac{\delta_{ij}}{3} - \frac{B_i B_j}{B^2} \right) \left( B^{-2} B \cdot B \cdot \nabla V - \frac{V \cdot V}{3} \right),
\]

where \( B \) is magnetic field, \( V \) is the plasma velocity, \( \delta_{ij} \) is the Kronecker delta function, and \( \eta_0 \) is a viscosity coefficient given by

\[
\eta_0 = 0.96 \rho_p \tau_p,
\]

where \( \rho_p \) is the proton pressure. Using equation (1),

\[
\eta_0 = 10^{-16} T_p^{5/2} \text{ g cm}^{-1} \text{ s}^{-1}
\]

for a Coulomb logarithm of 22. Equation (2) is a covariant form (Lifshitz and Pitaevskii 1981) of a result given by Braginskii (1965). It ignores terms associated with four other viscosity coefficients, two of which are smaller than \( \eta_0 \) by \( (\omega_{cp} \tau_p)^{-1} \) and two of which are smaller by \( (\omega_{cp} \tau_p)^{-2} \).

If \( B \) is in the \( z \)-direction, then

\[
S_{xx} = S_{yy} = -\eta_0 \left( \frac{V \cdot V}{3} - \frac{\partial V_z}{\partial z} \right),
\]

and

\[
S_{zz} = -2S_{xx}.
\]

All off-diagonal \( S_{ij} \) are zero. The volumetric force due to viscosity is the negative of the divergence of \( S \). To see the effect of \( S \), ignore for the moment all other forces in the momentum equation and take \( \eta_0 = \text{constant} \). It is then readily shown that equations (4) and (5) lead to

\[
\frac{\rho}{\eta_0} \frac{dD}{dt} = \frac{1}{3} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4 \frac{\partial^2}{\partial z^2} \right) D,
\]

where \( \rho \) is mass density,

\[
D \equiv V \cdot V/3 - \frac{\partial V_z}{\partial z}.
\]
and \( \frac{d}{dt} = \frac{\partial}{\partial t} + V \cdot \nabla \). Evidently, viscosity tends to smooth out variations of \( D \) via the anisotropic diffusion term on the right-hand side of equation (6). The diffusive time scale is

\[
\tau_v \approx L^2 \rho/\eta_0, \tag{8}
\]

where \( L \) is a characteristic length scale. Coronal active regions are highly structured on scales of a few arc seconds or smaller, and we might conservatively take \( L = 2000 \) km. If \( \rho = 5 \times 10^{-15} \) g \( \text{cm}^{-3} \) and \( T_e = 2.5 \times 10^6 \) K, then \( \tau_v \approx 200 \) s. This is comparable to observed active region coronal time scales (e.g., Withbroe, Habal, and Ronan 1985), and we surmise that viscosity may be important in the corona.

The volumetric heating rate due to viscosity is

\[
Q_v = -\nabla_j \frac{\partial V_j}{\partial x_i}. \tag{9}
\]

Using equation (2), we find

\[
Q_v = 3\eta_0 (B^{-2} B \cdot B \cdot \nabla V - \nabla \cdot (V/3))^2. \tag{10}
\]

Note that \( Q_v \) is positive definite. If \( B \) is in the \( z \)-direction, then

\[
Q_v = 3\eta_0 D^2. \tag{11}
\]

It is possible to find flows where there is viscous heating \( (D \neq 0) \) but where viscosity does not affect the momentum equation (which involves only derivatives of \( D \)). The reconnection-type flow of Sonnerup and Priest (1975) is one example of such a flow. They took

\[
V = k_1 (-x \hat{x} + z \hat{z}), \tag{12}
\]

where \( k_1 \) is an arbitrary constant. Then

\[
D = -k_1. \tag{13}
\]

There is viscous heating, but viscosity drops out of the momentum equation because the viscous stresses are divergence-free. The viscous heating is balanced energetically by the unbalanced viscous stresses at the boundaries of the system.

Viscous heating can be much larger than Joule heating. Joule heating is

\[
Q_j = j^2/\sigma, \tag{14}
\]

where \( j \) is current density and \( \sigma \) is resistivity. To estimate \( Q_j \), take

\[
j \approx (e/4\pi)(\Delta B/L), \tag{15}
\]

and

\[
Q_j \approx \frac{(\Delta B)^2 \chi}{4\pi L^2}, \tag{16}
\]

where \( \chi = c^2/(4\pi \sigma) \) is the magnetic diffusivity and \( \Delta B \) is the magnetic field change over the distance \( L \). If

\[
Q_v \approx \eta_0 V^2/L^2, \tag{17}
\]

we have

\[
\frac{Q_v}{Q_j} \approx \frac{4\pi \eta_0 V^2}{(\Delta B)^2 \chi}. \tag{18}
\]

Numerically,

\[
\chi \approx 2 \times 10^{13} T_e^{-3/2} \text{ cm}^2 \text{ s}^{-1}, \tag{19}
\]

where \( T_e \) is electron temperature, the Coulomb logarithm has been taken to be 22, and we have used the perpendicular resistivity. Thus

\[
Q_v/Q_j \approx 6.3 \times 10^7 T_e^2 V_6^2 (\Delta B)^{-2}, \tag{20}
\]

where \( T_6 \) is temperature (assumed equal for electrons and protons) in units of \( 10^6 \) K, and \( V_6 \) is velocity in units of 10 km s\(^{-1}\). For typical coronal values, i.e., \( V_6 \approx (1-3), T_6 \approx (1-3) \), and \( \Delta B \approx (1-100) \) G, we find \( Q_v \gg Q_j \). We will find below, however, that the diffusion region in magnetic reconnection is an important exception.

Viscous heating can be comparable to heating by electron heat conduction. The latter is

\[
Q_e = \kappa_\parallel \left( \frac{\partial T_e}{\partial \zeta} \right)^2 T_e^{-1}, \tag{21}
\]

where \( \kappa_\parallel \) is the heat conductivity along \( B \):

\[
\kappa_\parallel = 8.4 \times 10^{-7} T_e^{5/2} \text{ (cgs)} \tag{22}
\]
for a Coulomb logarithm of 22. The temperature gradient $\partial T/\partial z$ is difficult to estimate in general. For linear long-period waves the temperature fluctuations can be approximately adiabatic. Then

$$\frac{\Delta T}{T_e} \approx (\gamma - 1) \frac{V}{V_{ph}},$$

(23)

where $V_{ph}$ is the wave phase speed and $\gamma$ is the ratio of specific heats. Thus

$$\frac{Q_{cd}}{Q_e} \approx 1.2(\gamma - 1)^{-2} V_{ph,s}^2 T_e^{-1},$$

(24)

where $V_{ph,s}$ is $V_{ph}$ in units of 1000 km s$^{-1}$. If $V_{ph}$ is the Alfvén speed, then $Q_{cd} \approx Q_e$ in the corona.

In spite of its potential importance, there have been relatively few studies of the effects of coronal viscosity. Gordon and Hollweg (1983) and Steinolfson and Priest (1986) have looked at viscous damping of coronal surface waves. They conclude that viscous damping of coronal surface waves can heat the corona only if $B$ is of the order of a few gauss, which is probably not the case. (Gordon and Hollweg also considered wave damping via electron heat conduction and came to the same conclusion.) Tachih, Steinolfson, and Van Hoven (1985) have shown that viscosity can reduce the growth rate of the tearing mode, but their work has two defects: they use the form of the viscous stress tensor appropriate to very weak magnetic fields (i.e., $\alpha T_p \tau_p \ll 1$), and they take the smallest of the five viscosity coefficients given by Braginskii (1965) rather than $\eta_0$, Nocera, Priest, and Hollweg (1986) have considered the propagation of shear Alfvén waves, including their nonlinear coupling into the compressive fast and slow modes. Their analysis properly includes the viscous stress given by equation (2), and they show how viscosity can lead to an effective nonlinear dissipation.

In this paper we wish to discuss the physical basis of equation (2). We will provide a derivation of equation (2) which illuminates the underlying physics, but which does not give the numerical value of $\eta_e$ exactly. Our derivation will show that the stress tensor can contain other terms in addition to those given in equation (2). We will discuss the importance of those terms in the solar corona, and thus assess the validity of equation (2) in the coronal context. We will find that equation (2) is usually valid, except in the diffusion region of magnetic reconnection.

Our derivation will also allow us to make some statements about the electron stress tensor. In the classical analysis of Braginskii (1965), an expression is given for the electron viscous stress tensor, which is identical to equation (2), except that $\eta_0$ must be replaced by

$$\eta_e = 0.73 p_e \tau_e,$$

(25)

where $p_e$ is the electron pressure and

$$\tau_e = 1.3 \times 10^{-2} T_e^{3/2}/n$$

(26)

for a Coulomb logarithm of 22. Thus $\eta_e/\eta_0 \approx 1.3 \times 10^{-2}$. We will find that Braginskii’s expression for the electron viscosity may well be invalid in the corona. But it is probably still true that $S_{ij}$ for the electrons is small compared to $S_{ij}$ for the protons.

Our present work is an extension of several recent papers on viscosity in a magnetized plasma. Bravenec, Berk, and Hammer (1982) used a variation of the CGL equation (Chew, Goldberger, and Low 1956) to derive equation (2) for the special case of field-aligned flow in a multiple mirror configuration. Hollweg (1985) extended their analysis to give the fully covariant form of equation (2). These papers show that the viscous stress tensor, as given by equation (2), is the result of the plasma’s tendency to develop small field-aligned thermal anisotropies as it evolves. Coulomb collisions oppose the production of anisotropy and tend to distribute changes of internal energy through all three degrees of freedom; it is this latter aspect which ultimately leads to irreversible heating, equation (10). Equation (2) is associated with a gyrotropic pressure tensor of the form of equation (43) below; off-diagonal terms in the pressure tensor do not contribute to equation (2). These two papers have the disadvantage of beginning with the CGL equations which assume (among other things) frozen-in magnetic field:

$$\frac{\partial B}{\partial t} = \nabla \times (V \times B);$$

(27)

the following intermediate result is then derived:

$$S_{ij} = 3\eta_0 \left( \frac{\delta_{ij}}{3} - \frac{B_i B_j}{B^2} \right) \left( \frac{d \ln B}{dt} - \frac{2}{3} \frac{d \ln \rho}{dt} \right).$$

(28)

Equation (28) can be found also in Rossi and Olbert (1970). Equation (27) is then used again to obtain equation (2). This procedure leaves open the question of what happens if the field is not frozen-in, as in reconnection for example.

This question has been answered in part by a recent paper of Holzer, Leer, and Zhao (1986). They too consider the physical basis of viscosity, but in the solar wind context. Subject to a number of simplifying assumptions (steady, spherically symmetric, radial flow with a spiral magnetic field which is frozen into the flow), they too show that the viscous stress tensor is associated with the development of field-aligned thermal anisotropies. But their derivation begins with a more fundamental set of equations than the CGL equations, and careful analysis reveals that the frozen-in assumption is not essential to their derivation. It can be concluded, therefore, that equation (2) can be valid even when the magnetic field is not frozen into the flow, while equation (28) is then no longer valid. This will be demonstrated below in § IV.

The goal of this paper is to extend the analyses of Hollweg (1985) and Holzer, Leer, and Zhao (1986) in several respects: (1) We
II. MOMENTS OF THE BOLTZMANN EQUATION

Begin with the Boltzmann equation for any plasma species:

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + a \cdot \frac{\delta f}{\delta v} = \frac{\delta f}{\delta t}. \tag{29}$$

The distribution function $f$ is a function of the seven-dimensional space consisting of configuration space, velocity space, and time; the seven coordinates are taken to be independent. The acceleration $a$ includes electric, magnetic, and gravitational fields. The term $\delta f/\delta t$ contains Coulomb collisions and any other effects (such as wave-particle interactions) not included in $a$.

Multiply equation (29) by $\psi = \psi(r, v, t)$ and integrate over the three velocity coordinates. (This is a variation on taking the usual moments of the Boltzmann equation, where it is assumed that $\psi$ is a function only of $v$.) After defining

$$n\langle \psi \rangle = \int \psi f dv,$$

we obtain

$$\frac{\partial}{\partial t} n\langle \psi \rangle - n\left( \frac{\partial \psi}{\partial t} \right) + \nabla \cdot n\langle \psi v \rangle - n\langle v \cdot \nabla \psi \rangle - n\left( \frac{qE}{m} + g \right) \cdot \frac{\partial \psi}{\partial v} - \frac{qn}{mc} \left( v \times B \right) \cdot \frac{\partial \psi}{\partial v} = \int \psi \frac{\delta f}{\delta t} dv. \tag{31}$$

In equation (31) $q$ is charge, $m$ is mass, $E$ and $q$ are electric and gravitational fields, and $c$ is the speed of light; it has been assumed that $f \to 0$ very rapidly as $|v| \to \infty$.

Take now

$$\psi = \frac{1}{2} m v_\|^2, \tag{32}$$

where

$$v_\| = v \cdot \hat{b} \tag{33}$$

and $\hat{b} \equiv B/|B|$. Note that $\psi$ depends on $r$ and $t$ through $\hat{b}$. We have also

$$\frac{\partial \psi}{\partial t} = m v_\| \frac{\partial \hat{b}}{\partial t}, \tag{34}$$

$$\frac{\partial \psi}{\partial v} = m v_\| \hat{b}, \tag{35}$$

and

$$\nabla \psi = m v_\| (v \cdot \nabla \hat{b} + v \times \nabla \times \hat{b}). \tag{36}$$

It is convenient to define

$$w \equiv v - V, \tag{37}$$

where $V = \langle v \rangle$. It is also convenient to define a “parallel pressure”

$$p_\| \equiv \rho \langle w_\|^2 \rangle, \tag{38}$$

and a “parallel heat flux”

$$q_\| \equiv \rho \langle w_\|^2 w \rangle / 2. \tag{39}$$

Equations (32)–(39) are inserted into equation (31). The resulting equation can be simplified by combining it with the equation for mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = \int m \frac{\delta f}{\delta t} dv, \tag{40}$$

and with

$$\rho \frac{dV_\|}{dt} - pV_\| \cdot \nabla V_\| = -V_\| (V \cdot p) \cdot \hat{b} + nV_\| (qE_\| + mg_\|) + mV_\| \int w_\| \frac{\delta f}{\delta t} dv, \tag{41}$$

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where \( \mathbf{p} \) is the pressure tensor. Equation (41) has been obtained by taking the scalar product of \( V_{\parallel} \mathbf{b} \) with the equation for momentum conservation. There results

\[
\frac{1}{2} \frac{\partial p_{\perp}}{\partial t} - \rho \langle w_{\parallel} w \rangle \cdot \frac{\mathrm{d} \mathbf{b}}{\mathrm{d} t} + \mathbf{V} \cdot \left( \frac{1}{2} p_{\parallel} V + q_{\parallel} \rho V_{\parallel} \langle w_{\parallel} w \rangle \right) - \rho \langle w_{\parallel} w + w_{\parallel} V \rangle \cdot (\mathbf{w} \cdot \nabla \mathbf{b}) - V_{\parallel} \cdot (\mathbf{p} \cdot \mathbf{b}) = \int \frac{1}{2} m w_{\parallel} \frac{\partial f}{\partial t} \, d^3v .
\]

Equation (42) is quite general. Further simplification results from taking a gyrotropic pressure tensor

\[
p_{ij} = p_{\perp} \delta_{ij} + (p_{\parallel} - p_{\perp}) b_i b_j .
\]

Equation (43) will be a good approximation if the gyrofrequency, \( \omega_g \), is much larger than any rate at which the plasma evolves, since \( \omega_g \) is a measure of the rate at which deviations from gyrotropy are wiped out. Equation (42) then becomes

\[
\frac{1}{2} \frac{d p_{\parallel}}{d t} + \frac{1}{2} p_{\parallel} \mathbf{V} \cdot \mathbf{V} + \mathbf{V} \cdot \mathbf{q}_{\parallel} + p_{\parallel} \mathbf{b} \cdot \mathbf{b} \cdot \nabla \mathbf{V} - \rho \langle w_{\parallel} w \mathbf{w} \cdot \mathbf{w} \cdot \nabla \mathbf{b} \rangle = \int \frac{1}{2} m w_{\parallel} \frac{\partial f}{\partial t} \, d^3v ,
\]

where we have used \( \mathbf{b} \cdot \frac{d \mathbf{b}}{d t} = 0 \) since \( \mathbf{b} \) is a unit vector. Equation (44) is the desired equation for \( p_{\parallel} \). We will later show, in § IV, how equation (44) can be further manipulated into one of the CGL equations.

We now want an equation for \( p_{\perp} \). This is most easily obtained from the full thermal energy equation, which is well known:

\[
\frac{d}{d t} \frac{1}{2} \text{Tr} \mathbf{p} + \frac{1}{2} \text{Tr} \mathbf{p} \mathbf{V} \cdot \mathbf{V} + p_{\parallel} \frac{\partial V_{\parallel}}{\partial x_i} = -\mathbf{V} \cdot \mathbf{q} + \frac{1}{2} \int w_{\parallel} \frac{\partial f}{\partial t} \, d^3v ,
\]

where \( \text{Tr} \) is the trace and \( \mathbf{q} \) is the total heat flux:

\[
\mathbf{q} \equiv \rho \langle w^2 w \rangle / 2 .
\]

Combining equations (43)-(45) yield

\[
\frac{d p_{\perp}}{d t} + 2 p_{\perp} \mathbf{V} \cdot \mathbf{V} - p_{\perp} \mathbf{b} \cdot \mathbf{b} \cdot \nabla \mathbf{V} = -\mathbf{V} \cdot \mathbf{q}_{\perp} - \rho \langle w_{\parallel} w \mathbf{w} \cdot \mathbf{w} \cdot \nabla \mathbf{b} \rangle + \frac{1}{2} m \int w_{\parallel}^2 \frac{\partial f}{\partial t} \, d^3v ,
\]

where \( w_{\perp}^2 = w^2 - w_{\parallel}^2 \) and \( \mathbf{q}_{\perp} = \mathbf{q} - \mathbf{q}_{\parallel} \). We will later show, in § IV, how equation (47) can be manipulated into one of the CGL equations.

We now want an equation for the anisotropy, \( p_{\perp} - p_{\perp} \). We first combine equations (44)-(47) to yield

\[
\frac{d}{d t} \ln \frac{p_{\perp}}{p_{\parallel}} + \mathbf{V} \cdot \mathbf{V} - 3 \mathbf{b} \cdot \mathbf{b} \cdot \nabla \mathbf{V} = -p_{\perp}^{-1} \mathbf{V} \cdot \mathbf{q}_{\perp} + 2p_{\parallel}^{-1} \mathbf{V} \cdot \mathbf{q}_{\parallel} - 3q_{\parallel} \cdot \nabla \mathbf{b} (p_{\perp}^{-1} + 2p_{\parallel}^{-1}) + \frac{m}{2p_{\perp}} \int w_{\parallel}^2 \frac{\partial f}{\partial t} \, d^3v - \frac{m}{p_{\parallel}} \int w_{\parallel}^2 \frac{\partial f}{\partial t} \, d^3v ,
\]

where

\[
q_{\parallel} \cdot \nabla \mathbf{b} = \rho \langle w_{\parallel} w \mathbf{w} \cdot \mathbf{w} \cdot \nabla \mathbf{b} \rangle .
\]

We now split the two integrals on the right-hand side of equation (48) into self-collisions plus everything else. We further make the Ansatz that the self-collisions lead to a linear relaxation of the anisotropy while conserving internal energy. Thus

\[
\frac{1}{2} \int w_{\parallel}^2 \frac{\partial f}{\partial t} \, d^3v = \frac{\nu (p_{\parallel} - p_{\perp})}{3} + 2Q_{\perp} ,
\]

\[
\frac{1}{2} \int w_{\parallel}^2 \frac{\partial f}{\partial t} \, d^3v = \frac{\nu (p_{\parallel} - p_{\perp})}{3} + Q_{\parallel} .
\]

Here \( \nu \) is the rate at which the anisotropy relaxes due to self-collisions, and the Q's represent everything else (such as collisions with other species or wave-particle interactions). Equation (48) then becomes

\[
\nu (p_{\parallel} - p_{\perp}) = p_{\parallel} p_{\perp}^{-1} \left( \mathbf{V} \cdot \mathbf{V} - 3 \mathbf{b} \cdot \mathbf{b} \cdot \nabla \mathbf{V} + \frac{d}{d t} \ln \frac{p_{\perp}}{p_{\parallel}} \right) + 3q_{\perp} \cdot \nabla \mathbf{b} + 2p_{\perp} p_{\parallel}^{-1} (Q_{\parallel} - \mathbf{V} \cdot \mathbf{q}_{\parallel}) - p_{\parallel} p_{\perp}^{-1} (2Q_{\perp} - \mathbf{V} \cdot \mathbf{q}_{\perp}) ,
\]

where \( p = (p_{\parallel} + 2p_{\perp})/3 \). Equation (51) is very similar to equation (17) of Holzer, Leer, and Zhao (1986), but those authors did not explicitly give the terms involving the Q's and \( q \)'s, and they did not use a covariant form.

Equation (51) is the basis of our discussion of viscosity and its validity, which we take up in the next section.

### III. VISCOSITY AND THE VALIDITY

Equation (43) can be rewritten as

\[
p_{ij} = p \delta_{ij} + S_{ij} ,
\]

where

\[
S_{ij} = (p_{\parallel} - p_{\perp}) (b_i b_j - \delta_{ij} / 3) .
\]
The quantity $S_{ij}$ is a generalized stress tensor which can be calculated if $p_{||} - p_{\perp}$ is given by equation (51). Note that $S_{ij}$ involves terms which are not derivatives of the velocity, and it is in general not proper to call it a viscous stress tensor.

When does $S_{ij}$ agree with the classical viscous stress tensor, viz. equation (2)? Comparing equations (53) and (51) with equation (2) shows that this is the case when three conditions are satisfied: (1) self-collisions are so frequent that $p_{||} \approx p_{\perp} \approx p$; (2) the $q$-terms in equation (51) are negligible. (3) the $Q$-terms in equation (51) are negligible. In addition, we have to make the identification

$$\eta_0 = \frac{p}{v}$$

(54)

since our analysis does not yield the value of $\eta_0$ exactly; thus for protons

$$v^{-1} = 0.96 \tau_p.$$  

(55)

We will first consider protons in the solar corona, and the extent to which these three conditions are satisfied. We will consider a few representative examples, but we caution the reader that these conditions should be reexamined for any specific case at hand.

**a) Protons**

The first condition requires $v \gg V/L$, or $\tau_p \ll L/V$. In an active region loop we might have $L \approx 2000$ km, $V \approx 20$ km s$^{-1}$, and $\tau_p \approx 1$ s (for $T_p = 2.5 \times 10^9$ K and $n = 3 \times 10^9$ cm$^{-3}$). The condition is reasonably well satisfied. Near the base of a coronal hole we might have $L \approx 5000$ km, $V \approx 20$ km s$^{-1}$, and $\tau_p \approx 3.75$ s (for $T_p = 10^6$ K and $n = 2 \times 10^9$ cm$^{-3}$), and the condition is again satisfied.

The second condition roughly requires

$$\nabla \cdot \mathbf{V} \cdot q_p \ll pV/L.$$  

(56)

To estimate $\nabla \cdot q_p$, we shall assume that $q_p$ and $\mathbf{q}_e$ are given by the classical expressions (Braginskii 1965), and that $T_e = T_0$. Then

$$\nabla \cdot q_p = 0.04 V \cdot q_e.$$  

To estimate $\nabla \cdot q_e$ we shall assume that $\nabla \cdot q_e$ roughly balances the coronal radiative losses, $n^2 P(T)$. Inequality (56) becomes then

$$2.9 \times 10^{19} P \ll V_0 T_0 L_0^{-1} n_0^{-1},$$  

(57)

where $V_0$ is in units of $10^9$ cm s$^{-1}$, etc. At coronal temperatures ($T_0 > 1$) we have $P \lesssim 10^{-22}$ ergs cm$^{-3}$ s$^{-1}$ (e.g., Fig. 10 of Rosner, Tucker, and Vaiana 1978), and inequality (57) would appear to be well satisfied for the coronal conditions we have heretofore assuming.

Inequality (57) can be violated in the chromosphere-corona transition region, however, where $T \approx 10^5$ K and $P \approx 6 \times 10^{-22}$ ergs cm$^{-3}$ s$^{-1}$. For example, in the transition region below an active region loop we might have $T_0 \approx 0.1$ and $n_0 = 75$ (the electron and proton pressures are then 1 dyne cm$^{-2}$). Taking $V_0 = 2$ and $L_0 = 2$ then gives the result that the left-hand side of equation (57) exceeds the right-hand side by one order of magnitude. In the transition region below a coronal hole we might have $V_0 = 2$, $L_0 = 5$, $T_0 = 0.1$, and $n_0 = 2$, and the left- and right-hand sides of equation (57) are then comparable. Thus the classical expression for viscosity can fail in the transition region. However, $v$ becomes very large in the transition region ($v \propto T^{-5/2}$ if the pressure is held constant), and $S_{ij}$ will probably be unimportant there anyway.

Consider now that third condition above, that the $Q$'s be negligible compared to the velocity gradient terms in equation (51). This is difficult to assess quantitatively, since the $Q$'s might well contain the unknown source of coronal heating itself, e.g., wave-particle interactions. However, we can make a few comments of a general nature.

Suppose the protons are heated viscously, perhaps via viscous dissipation of MHD waves, or via a viscous dissipation range in MHD turbulence. In effect, this occurs via the self-collisions which irreversibly distribute internal energy changes among the three degrees of freedom. Viscous heating of the protons would make them hotter than the electrons. Coulomb collisions with the electrons would tend to cool the protons. This electron-proton coupling is represented by the $Q$'s in equation (51). Now we expect that the energy lost to the electrons from each proton degree of freedom will be proportional to the energy in that degree of freedom. Thus

$$\frac{Q_{||}}{p_{||}} = \frac{Q_{\perp}}{p_{\perp}} = \frac{Q}{p}.$$  

(58)

In that case the $Q$ terms in equation (51) sum to zero, and do not modify the classical result for the viscous stress.

On the other hand, suppose the electrons are directly heated, via wave-particle interactions or via Joule heating. They would then heat the protons via Coulomb collisions. This proton heating is represented by the $Q$'s in equation (51). In this case we expect that the three proton degrees of freedom are heated equally, so

$$Q_{||} = Q_{\perp} = Q.$$  

(59)

In that case (51) becomes

$$\left(1 + \frac{2Q}{pv}\right)p_{||} - p_{\perp} = \frac{p_{||} + p_{\perp}}{p} \left(\nabla \cdot V - 36 \cdot \delta \cdot \nabla V + \frac{d}{dt} \ln \frac{p_{||}}{p_{||}}\right),$$  

(60)

if we ignore the $q$ terms. Isotropic proton heating reduces $p_{||} - p_{\perp}$, and the effective viscous stress, by a factor $(1 + 2Q/pv)^{-1}$. Using equations (55) and (1), we find

$$\frac{2Q}{pv} = 10QT_0^{1/2}n_0^{-2}.$$  

(61)
As a rough estimate, we assume that the proton heating is the same as the electron heating, which is in turn comparable to the radiative losses; thus,

$$Q \approx n^3 P(T)/3,$$

(62)

where the factor 3 refers to the three degrees of freedom. Then

$$\frac{2Q}{p v} \approx 3 \times 10^{18} P(T) T_6^{1/2}.$$

(63)

Since $P(T) < 6 \times 10^{-22}$ ergs cm$^{-3}$ s$^{-1}$, we conclude that $2Q/pv \ll 1$. Again, the classical expression for the viscous stress is valid.

So far our numerical estimates have used reasonable numbers for average coronal values. A different set of numbers would result if the coronal heating and accompanying flows were concentrated into small regions of intense heating. This would be the case if the corona were heated via magnetic reconnection, as proposed by Parker (1983), for example. The plasma is heated by Joule heating in the “diffusion region,” the energy being derived from annihilation of the magnetic field. The diffusion region is of very small size, however, implying large velocity gradients and possibly large viscous effects. We will now investigate the importance of the viscous effects, and the validity of the classical description, in the simple model of the diffusion region discussed by Sonnerup and Priest (1975). We will find, perhaps surprisingly, that viscosity is unimportant in the diffusion region.

Sonnerup and Priest (1975) considered the case where

$$B = B(x) \hat{z},$$

(64)

and

$$V = k_1(-x \hat{x} + z \hat{z}),$$

(65)

where $k_1$ is a constant. Thus $V \cdot V = 0$, and the volume force due to viscosity vanishes. The electric current is given by Ohm’s law

$$j_y = \sigma E_y + k_1 x B/c,$$

(66)

where $E_y$ is a uniform electric field. From Ampere’s law we have

$$\frac{c}{4\pi\sigma} \frac{dB}{dx} = - \frac{k_1 x B}{c},$$

(67)

where $E \equiv -E_y$. The solution to equation (67) has been given by Sonnerup and Priest for the boundary condition $B(0) = 0$:

$$B = E \left( \frac{4\pi\sigma}{k_1} \right)^{1/2} e^{-\xi^2/2} \int_0^\xi e^{u^2/2} du,$$

(68)

where

$$\xi = x/k_1 \sqrt{1}.$$  

(69)

The solution (68) reaches a maximum near $|\xi| = 1$. Roughly speaking, $|\xi| < 1$ is the diffusion region where magnetic energy is being annihilated. The Joule heating throughout the diffusion region is reasonably well approximated by its value at $x = 0$. From equations (66), (68), and (69), we find

$$(j^2/\sigma)_0 \approx 4(B_i^2/8\pi) t_r^{-1},$$

(70)

where $B_i = B(|\xi| = 1)$, i.e., the magnetic field at the boundary of the diffusion region, and $t_r \equiv x(|\xi| = 1)/V(|\xi| = 1, z = 0) = k_1^{-1}$ is a measure of the time it takes a parcel of plasma to transit the diffusion region.

The viscous heating for this flow is given by equations (11) and (13), if we assume for the moment that the classical expression is valid. Thus

$$\frac{Q_x}{Q_j} = 0.7 p_p \left( \frac{B_i^2}{8\pi} \right)^{-1} \tau_p t_r^{-1},$$

(71)

where we have made use of equation (3a). Now in the corona we usually have $p \ll B^2/8\pi$, but in the diffusion region, where the field is annihilated, $p_p$ could become large enough to balance $B_i^2/(8\pi)$. Thus $Q_x/Q_j \lessapprox \tau_p t_r^{-1}$. But we have used the classical expressions for the viscous heating which require $\tau_p \ll t_r$. Thus we obtain the somewhat surprising result that $Q_x \ll Q_j$ in the diffusion region, if the viscous stress is classical.

Incidentally, we can show that $2Q/pv \ll 1$ for this case. Suppose that the protons share the Joule heating, via Coulomb coupling with the electrons. Then $Q = (3/4)(j^2/\sigma)/3$, the factor 3 again referring to the three degrees of freedom; this is probably an overestimate since the electron-proton coupling is weak. Using equation (70) then gives

$$\frac{2Q}{pv} = \frac{4}{3} \left( \frac{B_i^2}{8\pi p_p} \right) (v t_r)^{-1}.$$

(72)

Since $(v t_r) \gg 1$ by assumption, we find that $2Q/pv$ makes only a small modification to the classical expression for the viscous stress.

By the same token, we can argue that the $q$ terms in equation (51) are unimportant in the diffusion region. We again take
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\( \mathbf{V} \cdot \mathbf{q}_p = 0.04 \mathbf{V} \cdot \mathbf{q}_e \). In order of magnitude we might have \( \mathbf{V} \cdot \mathbf{q}_e \approx j^2/\sigma \). The \( q \) terms will then be small compared to the velocity gradient terms in equation (51) if

\[
\frac{B^2}{8\pi p} \ll 19 .
\]  

(73)

Since we expect the left-hand side of equation (73) to be close to unity in the diffusion region, we conclude that the \( q \) terms in equation (51) can be neglected.

Summing up then, if \( v_{tr} \gg 1 \) in the diffusion region, then the classical expression for the viscous stress is valid, but \( Q_e \ll Q_j \).

On the other hand, the diffusion region is usually very small, and it is more likely that \( v_{tr} \ll 1 \). In that case the classical approximation for the viscous stress fails, and it becomes necessary to solve equations (44), (47), and (51) for \( p_\parallel \) and \( p_\perp \). For example, if we drop the \( q \) terms and take \( v = 0 \), equation (60) becomes

\[
\frac{p_\parallel - p_\perp}{p} = \frac{p_\parallel}{2Qp} \left( \frac{d}{dt} \ln p_\parallel - \frac{3}{p_\parallel} \right) ,
\]

(74)

if \( V \) is given by equation (65). Even if we assume that the protons share the Joule heating, so that \( Q = j^2/6\sigma \), we can conclude from equation (74) that \( p_\parallel - p_\perp \) will not be small if \( p \approx B^2/8\pi \) in the diffusion region. Similarly, we can conclude that the heating term, \(-p_\parallel \partial V_\parallel/\partial x_\parallel \), will not be small compared to the Joule heating. Solutions for the diffusion region such as have been given by Sonnerup and Priest (1975) will then have to be modified.

This concludes our discussion of the relationship between the generalized stress tensor \( S_{ij} \), and the classical viscous stress tensor given by equation (2), for the protons. We will now offer some similar comments about the electrons.

\( b) \) Electrons

Braginskii (1965) gives a form for the electron viscous stress tensor which is identical to equation (2), except that

\[
\eta_e = 0.73 p_e \tau_e ,
\]

(75)

where \( \tau_e \) is the electron collision time,

\[
\tau_e = \tau_p/60 .
\]

(76)

Thus if equation (51) is taken to refer to the electrons, we must make the identification

\[
\nu^{-1} = 0.73 \tau_e ,
\]

(77)

(see eq. [55]).

In order for \( S_{ij} \) to agree with the classical expression for the viscous stress tensor, the same three conditions given above (following eq. [53]) must be satisfied. The first condition, the \( p_\parallel \approx p_\perp \approx p \), is usually satisfied, since \( \tau_e \) is small. The other two conditions, that the \( q \) terms and \( Q \) terms in equation (51) be small, can be violated for the electrons. For example, the smallness of the \( q \) terms roughly requires

\[
\mathbf{V} \cdot \mathbf{q}_e \ll pV/L .
\]

(77)

If \( \mathbf{V} \cdot \mathbf{q}_e \) balances the radiative losses, equation (77) becomes

\[
7.2 \times 10^{20} P \ll V_e T_e L_e^{-1} n_e^{-1} .
\]

(78)

This condition can be violated at temperatures below \( 10^6 \) K. Similarly, if \( \mathbf{V} \cdot \mathbf{q}_e \) balances the Joule heating in the diffusion region of reconnection, then equation (77) becomes

\[
\frac{B^2}{2\pi} \ll p_e ,
\]

(77a)

where equation (70) has been used; this condition too is violated. Thus the classical expression for the electron viscous stress can easily break down. However, in view of the smallness of \( \tau_e \), the stress tensor \( \delta_{ij} \) will probably be too small to be significant in the corona anyway.

This concludes our discussion of the relationship between the classical viscous stress tensor, equation (2), and the more general stress tensor given by equations (51) and (53). We will now take up another topic mentioned in the introduction, viz. the relationship between the classical viscous stress tensor and the alternate form given by equation (28), which is based on the CGL equations.

IV. THE CGL EQUATIONS

The induction equation is

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \mathbf{R} .
\]

(79)

Equation (79) can be regarded as defining \( \mathbf{R} \). In resistive MHD, \( \mathbf{R} \) will be the magnetic diffusion. By expanding the curl, we obtain

\[
\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \nabla \mathbf{V} - \mathbf{B} \cdot \nabla \mathbf{V} + \mathbf{R} .
\]

(80)
After combining equation (80) with equation (40), we obtain

$$\frac{d(B/\rho)}{dt} = \frac{(B \cdot \nabla V + R)}{\rho} - \frac{B}{\rho} \int \frac{\delta f}{\delta t} d^3v .$$  \hspace{1cm} (81)

Then taking the scalar product of equation (81) and \( \hat{b} \) yields

$$\frac{d \ln B/\rho}{dt} = \hat{b} \cdot \left( \frac{\hat{b} \cdot \nabla V + R}{\rho} \right) - \frac{m}{\rho} \int \frac{\delta f}{\delta t} d^3v .$$  \hspace{1cm} (82)

Equation (82) can be regarded as an equation for \( \hat{b} \cdot \hat{b} \cdot \nabla V \). Inserting this into equation (44), and again using equation (40), yields

$$\frac{d \ln \rho}{dt} \frac{B^2}{\rho^3} = \frac{2\hat{b} \cdot R}{B} + 2p_{\perp}^{-1} (q_3 \cdot \nabla \hat{b} - \nabla \cdot q_{\perp}) + \frac{m}{p_{\parallel}} \int \frac{\delta f}{\delta t} \left( \frac{w^2}{2} - \frac{3p_{\perp}}{\rho} \right) d^3v .$$  \hspace{1cm} (83)

Similarly, equation (47) yields

$$\frac{d \ln \rho}{dt} \frac{B^2}{\rho^3} = -\frac{\hat{b} \cdot R}{B} - \frac{p_{\perp}^{-1} (\nabla \cdot q_{\perp} - q_{\parallel} \cdot \nabla \hat{b})}{p_{\perp}} + \frac{m}{p_{\perp}} \int \frac{\delta f}{\delta t} \left( \frac{w^2}{2} - \frac{p_{\perp}}{\rho} \right) d^3v .$$  \hspace{1cm} (84)

Equations (83) and (84) are generalizations of equations (31) and (32) of Chew, Goldberger, and Low (1956). If the right-hand sides can be dropped, then they yield the familiar double-adiabatic equations of state:

$$\rho_{\perp}/(\rho B) = \text{constant} ;$$

and

$$\rho_{\parallel} B^2/\rho^3 = \text{constant} .$$

If we now use equations (50), equations (83) and (84) can be combined to give

$$\nu(p_{\parallel} - p_{\perp}) = \frac{\rho_{\parallel} p_{\perp}}{p} \left( \frac{d \ln \rho^2/\rho^3}{dt} + \frac{3\hat{b} \cdot R}{B} + \frac{d \ln \rho_{\parallel}}{dt} + 3q_{\perp} \cdot \nabla \hat{b} + 2p_{\perp}^{-1} (Q_{\parallel} - q_{\parallel}) - p_{\parallel}^{-1}(2Q_{\perp} - q_{\parallel}) \right) ,$$  \hspace{1cm} (85)

where we have made the additional assumption that

$$\int \frac{\delta f}{\delta t} d^3v = 0 .$$  \hspace{1cm} (86)

(This is the usual assumption that collisions do not create or destroy particles, but we have not used it until now.) Equation (85) is equivalent to equation (51). It can be used in the same way that equation (51) was used in the previous section.

The equation for the viscous stress tensor can be obtained by inserting equation (85) into equation (53), and making the further assumptions that the \( q \) terms and \( Q \) terms are negligible, and that \( p_{\parallel} \approx p_{\perp} \approx p \). There results

$$S_{ij} = 3\eta_0 \frac{\delta_{ij}}{3} - b_i b_j \left( \frac{d \ln B}{dt} - \frac{2 d \ln \rho}{dt} - \frac{\hat{b} \cdot R}{B} \right) ,$$  \hspace{1cm} (87)

where again \( \eta_0 = \nu/\nu \). This is the desired generalization of equation (28). Comparing the two expressions shows that equation (28) is appropriate only when \( R \) is negligible. Equation (28) would not be appropriate for the reconnection-type flow of Sonnerup and Priest (1975), for example. Thus equation (2) should be used for the viscous stress, or equation (87); equation (28) is suitable only when \( R \) can be neglected.

V. \textsc{summary}

Inspired by a paper by Bravenec, Berk, and Hammer (1982), Hollweg (1985) showed how the \( \eta_0 \) terms of Braginskii's (1965) viscous stress tensor could be derived (apart from a numerical factor) from the CGL equations. In essence, it was shown that a plasma will in general develop small thermal anisotropies as it evolves, and that these anisotropies fully account for the \( \eta_0 \) terms in the viscous stress tensor. This derivation made a number of simplifying assumptions, however. It was assumed that the magnetic field was frozen into the flow, and it was also assumed that heating terms due to heat conduction, wave-particle interactions, and inter species collisions, were small.

The goal of this paper has been to generalize the results of Hollweg (1985) by including the heating terms and by relaxing the frozen-in assumption. The only significant assumption is embodied in equation (50), which states that the self-collisions lead to a linear relaxation of the thermal anisotropy while conserving internal energy. The essential result is equation (51) for the anisotropy, which can be inserted into equation (53) to give a generalized stress tensor. This tensor contains terms which are not derivatives of the velocity, and it is in general not correct to call it a viscous stress tensor. Equation (51) was derived directly from the moments of the Boltzmann equation. This, of course, is a more fundamental starting point than the CGL equations used by Hollweg (1985) and Bravenec, Berk, and Hammer (1982). Equation (51) is very similar to equation (17) of Holzer, Leer, and Zhao (1986), who were studying the validity of the classical expression for the viscous stress in the solar wind. Our equation (51) is even more general than theirs, however.
We showed how equation (51) leads to the classical result (eq. [2]) when the q’s and Q’s are dropped, and when the self-collisions are so frequent that \( p_\parallel \approx p_\perp \approx p \). For average coronal conditions, we found that these requirements are probably satisfied for the protons. They can fail in the transition region, however, where \( T \approx 10^5 \) K, but \( p_\parallel - p_\perp \) will probably be very small and unimportant in the transition region anyway.

We also investigated the validity of equation (2) for the protons in the diffusion region of reconnection. If conditions are such that many self-collisions occur in the time it takes a plasma parcel to transit the diffusion region, then we found the surprising result that the viscous heating in the diffusion region will be small compared to the Joule heating there. On the other hand, the smallness of the diffusion region makes it unlikely that many collisions occur during one transit time. In that case the classical expression for viscosity fails, but the thermal anisotropy could be an important aspect of diffusion region dynamics and thermodynamics.

Regarding the electrons, we have shown that the classical expression for the electron viscosity can fail in the corona. But owing to the high rate of electron self-collisions, their thermal anisotropy will probably be small and unimportant anyway.

Finally, we have shown how the CGL-based derivations of Bravence, Berk, and Hammer (1982) and of Hollweg (1985) can be modified for the case when the magnetic field is not frozen into the flow. Equation (85) is the essential result in this case. And if the \( q \)’s and \( Q \)’s can be dropped, we obtain equation (87) as an alternate expression for the classical viscous stress tensor.

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