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LXXVI. Conditions for the Occurrence of Electrical Discharges in Astrophysical Systems

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Summary

Discharges are shown to be a possible source of high energy particles, if the current density is very large. The growth of the current density is discussed using the fact that the magnetic lines of force are approximately frozen into the ionized gas. It is shown that discharges are unlikely to occur anywhere except at neutral points of the magnetic field. Neutral points are found to be unstable in such a way that a small perturbation will start a discharge in a time of the order of the characteristic time of the system. Such discharges may account for aurorae, and may also occur in solar flares and the interstellar gas.

§ 1. Introduction

The possibility of the occurrence of electric discharges in astrophysical systems is important as an obvious source of high energy particles, if the accelerating voltage is large enough. The existence of large 'potential differences' in rotating magnetic stars has been pointed out by Alfvén and others (Alfvén 1950), but this is not a sufficient condition for a discharge to occur, as will be seen later. Further, the possibility of the particles in a small region reaching very high energies by absorbing energy from a large surrounding region is of more interest than the moderate heating of the material in a large region.

In the following no particular system is discussed, but any system of interest can be described for our purposes as a large mass of ionized gas in a more or less complicated state of motion. A 'discharge' will be a region in which the electrons are accelerated to high energies by the electric field, so that all the electrons are moving in the same direction with large velocities. If we suppose that the electrons acquire relativistic energies, the current density is then of order nec, where $n$ is the electron density. Now Maxwell's equations show that $c \text{curl} \mathbf{H}/4\pi$ must be approximately equal to the current density and $nec$ is found to be large when compared with the values of $c |\text{curl} \mathbf{H}|/4\pi$ usually expected in astrophysical systems. For instance in the chromosphere $n$ is about $10^{11}$ particles/cm$^3$ which requires $|\text{curl} \mathbf{H}| \sim 500$ gauss/cm and this is much

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larger than the values expected in sunspots. In the interstellar gas \( n \) is about one particle/cm\(^3\), and the interstellar magnetic field is generally supposed not to exceed \( 10^{-5} \) gauss; then in a discharge the field would have to change considerably in a distance of 20 metres. Consequently a discharge must be extremely thin in one direction. In order to examine how such a discharge can occur a general study of cosmic electrodynamics is required.

In this paper the conditions which can lead to the onset of a discharge will alone be discussed. The arguments to be used apply only when the current density is small compared with \( \kappa \); they are sufficient to determine the conditions under which the current density will grow, although the behaviour of the discharge, once started, is more complicated. For a discussion of the behaviour of discharges the reader is referred to Alfvén (1950).

\[ \text{§ 2. Cosmic Electrodynamics} \]

The fundamental variables of cosmic electrodynamics are the electric and magnetic fields, \( \mathbf{E} \) and \( \mathbf{H} \), the charge and current densities, \( \rho \) and \( \mathbf{j} \), the mass density \( \mu \), velocity \( \mathbf{u} \) and pressure \( p \) of the gas. Gaussian units are used. The fundamental equations are Maxwell's equations, the hydrodynamical equation

\[
\mu \mathbf{u} \equiv -\nabla p + \rho \mathbf{E} + \mathbf{j} \times \mathbf{H}/c, \quad \ldots \ldots \quad (1)
\]

in which

\[
\frac{d}{dt} \mathbf{u} = \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \cdot \nabla
\]

and Ohm’s law, which can be written (Sweet 1949)

\[
\mathbf{E} + \frac{(\mathbf{u} \times \mathbf{H})}{c} \equiv \frac{\mathbf{j}}{\sigma} + \frac{\mathbf{j} \times \mathbf{H}}{\kappa c}, \quad \ldots \ldots \quad (2)
\]

where \( \sigma \) is the conductivity and the last term is the Hall electric field; Cowling (1945) obtains \( \sigma = c^2 T^{3/2} / k \), where \( T \) is the temperature of the gas and \( k \) may be treated as a constant in these applications, equal to \( 6.8 \times 10^{13} \), so that

\[
\sigma = 1.3 \times 10^7 T^{3/2} \text{ sec}^{-1}.
\]

The natural approach to the problem by considering the effect of an ‘applied’ electric field is not convenient here owing to certain sources of confusion which will now be mentioned. Equation (2) is obtained by calculating \( \delta j / \delta t \) and should contain a term \( (m/\kappa e^2) \partial j / \partial t \) on the right-hand side, where \( m \) is the electron mass; using Maxwell’s equations this can be written as

\[
- \frac{m}{4\pi n e^2} (c^2 \text{ curl curl } \mathbf{E} + \frac{\partial^2 \mathbf{E}}{\partial t^2})
\]

\((mc^2/4\pi ne^2)^{1/2}\) is the ‘electron plasma wavelength’ and is usually small compared with the dimensions of astrophysical systems, so that the omission of if the effect plasma osc

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omission of this term from (2) is justified. It should be included, however, if the effect of an applied electric field is considered. It is then seen that plasma oscillations must occur. Nor should it be supposed that these oscillations are damped out and that the current density changes in such a way as to satisfy (2) with the initial value of \( E \). The situation is controlled by induction effects. Suppose that the magnetic field vanishes initially and the applied electric field is uniform: \( \text{curl } E \) vanishes, hence \( H \) continues to vanish, and \( \partial E / \partial t = -4\pi j = -4\pi \sigma E \). Consequently the electric field decays with a time constant \((4\pi \sigma)^{-1}\), or \(6 \times 10^{-9} T^{-3/2} \text{ sec}\).

The growth of the magnetic field and current density depends on \( \text{curl } E \) not vanishing as in the theory of the skin effect, and as this sort of behaviour is controlled by induction it is better not to consider an applied electric field, but to study the behaviour of the gas, when eqn. (2) is satisfied all the time. The values of the variables at any time must then have arisen from past developments of the system. The high conductivity makes it possible to find a more successful approach, and this has been developed by several authors. (Walén 1947, Elsasser 1947, Dungey 1950.) The expression for \( E \) given by eqn. (2) determines \( \partial H / \partial t \); the contribution of each term can be estimated and it is found that for a system of astrophysical dimensions a good approximation is obtained if both terms on the right-hand side are neglected. In this approximation the rate of change of the magnetic field is given by

\[
\frac{\partial H}{\partial t} = -(u \cdot \nabla)H + (H \cdot \nabla)u - H(\nabla \cdot u) \quad \ldots \ldots (3)
\]

and it can be shown that the magnetic flux, linked by a closed curve which moves with the gas, is a constant of the motion. It is then easy to see that the magnetic field is "frozen into" the gas. This approximation is used in the following discussion of the possible initiation of a discharge.

It may be recalled here that when the magnitude of the current density was compared with \( c \) \( \text{curl } H/4\pi \) in § 1, the displacement current was ignored. It can now be shown to be negligible. If the dimensions of the system are regarded as characterized by a length \( a \) and a velocity \( v \), its characteristic time will be \( a/v \) and we may then put for instance \( |\text{curl } H| \sim |H|/a \), \( |\partial E / \partial t| \sim v |E|/a \). Then since \( E \approx H \wedge u/c \), we have \( |\partial E / \partial t| \sim v^2 |H|/ac \) which can be neglected in comparison with \( c |\text{curl } H| \). Similarly in (1) \( |\rho E| \) is of order \( v^2 |H|/ac \) and can therefore be neglected in comparison with \( j \wedge H/c \). The electromagnetic force density can then be written (\( \text{curl } H \)) \( \wedge H/4\pi \) or \( (H \cdot \nabla)H - \frac{1}{2} \nabla |H|^2/4\pi \), and can be represented as a tension \( |H|^2/8\pi \) per unit area along the lines of force and a lateral pressure of the same amount perpendicular to them.

§ 3. DISCHARGES IN A MAGNETIC FIELD

Since \( |\text{curl } H| \) needs to be very large when there is a discharge, and since the lines of force of the magnetic field are frozen in, it is essential to start with some magnetic field, and investigate how \( |\text{curl } H| \) can grow.

3 B 2
Consider first the possibility of a discharge occurring anywhere other than at a neutral point of the magnetic field. If the current density were to have a large component perpendicular to the magnetic field the electromagnetic force density would be large. Such a discharge might be expected to occur, when a shock wave travels across a magnetic field, but shock waves will not be discussed in this paper.

If the current density is parallel to the magnetic field, the lines of force are twisted like the strands of a cable. Because of the effective tension of the electromagnetic force we expect them to resist this twisting. A simple illustration is obtained in cylindrical coordinates $\rho, \phi, z$ by taking $H_\rho=0$, $H_\phi=8\pi j/c$, $H_z=H$; where $H$ and $J$ are constants; then $j_\rho=j_\phi=0$, $j_z=J$. This represents a possible situation in the neighbourhood of a discharge, but it is necessary to consider what happens further away. The current lines must be closed and, unless they flow right round lines of force, they must cross the lines of force. Then there is a torque on the gas arising from the term $(H \cdot \nabla)H/4\pi$ in the force density, and directed so as to untwist the lines of force. Consequently the growth of the current density is opposed by the electromagnetic forces. This argument does not apply to the case when the current flows right round the lines of force, but then the lines of force are linked with each other a large number of times, if the current density is large. Now when lines of force are frozen into the gas, they cannot become linked during the course of the motion, and hence a large current density of this type cannot be a result of the motion. Consequently we do not expect discharges to occur except at neutral points of the magnetic field, which will now be discussed.

§ 4. Neutral Points

Giovanelli (1947, 1948) and Hoyle (1949) have suggested that the neighbourhood of a neutral point is the seat of a discharge. Giovanelli points out that solar flares frequently occur in positions where a neutral point of the sunspot field is expected, and Hoyle has suggested an explanation of the origin of the aurora involving the same idea. We distinguish between two types of neutral point: X-type as in fig. 1 (a), and O-type which occurs at the centre of O-shaped lines of force. The possibility of discharges occurring elsewhere was rejected because the electromagnetic forces oppose the growth of the current density, but at an X-type neutral point the opposite situation occurs.

The magnetic field in the neighbourhood of a neutral point is described by the tensor $\partial H_i/\partial x_j$. The antisymmetrical part relates to curl $H$ and therefore to $j$. Consider first the case when $j$ vanishes: $\partial H_i/\partial x_j$ then has principal axes which are orthogonal. Let these be taken as Cartesian axes, with the neutral point as origin. Then the field at a point on one of these axes has the direction of that axis. Also, since $\text{div } H=0$, the diagonal components of $\partial H_i/\partial x_j$ cannot all have the same sign. Let $\partial H_1/\partial x_1$ and $\partial H_2/\partial x_2$ have opposite signs. Now consider the field, when there is a current in the $z$-direction. The direction of the field at points in the $(x, y)$ plane lie for the case $\omega$ clockwise. The current density is the same at different positions of the principal axes because a small increase in $j$ reaches the point of force after a distance because the electric field of the discharge is large.

A steady state is the contribution of the current density.
(x, y) plane lies in the (x, y) plane. The lines of force are shown in fig. 1 (a), for the case when the direction of the field belonging to the current is clockwise. The principal axes are no longer perpendicular. The direction of the electromagnetic force is shown in fig. 1 (b) and the gas must flow in the same general direction; the gas will be stretched in the vertical direction in fig. 1. Since the lines of force are frozen into the gas, the principal axes will rotate towards each other. This suggests that the current density will be increased, in which case the situation is unstable, because a small current density will cause a motion which will in turn increase the current density. The current density will then grow until it reaches the proportions of a discharge. The approximation that the lines of force are frozen into the gas then breaks down near the neutral point because the electric field required to drive the current becomes important. A steady state will be reached when the decay of the current density due to the contribution of this accelerating field to \( \text{curl} \ \mathbf{E} \) balances the growth of the current density due to the motion.

Fig. 1

It remains to show that the current density does grow to the proportions of a discharge. In § 5 this result is proved omitting the effect of the pressure gradient; then, if it can be shown that the pressure gradient reinforces the electromagnetic force in the neighbourhood of the neutral point, the result holds a fortiori. In § 6 two-dimensional models are considered including the effect of the pressure gradient and it is shown that the condition of mechanical equilibrium requires the current density at an X-type neutral point to be infinite.
This conclusion that the current density will grow near a neutral point but not elsewhere, which appears to draw an absolute distinction between different points in space, may be further clarified if expressed in the following way. If there is a small disturbance in a region of non-zero magnetic field the current density will not become large in this region, but the disturbance will spread in the form of Alfvén waves; if, however, there is a neutral point anywhere a small disturbance can cause the current density to become large in the neighbourhood of the neutral point.

Fig. 2

(a)  (b)

The contribution of the accelerating electric field to $\partial H/\partial t$ can be pictured in terms of lines of force. If there are two lines of force as shown in fig. 2 (a), the direction of the current corresponds to a field in the clockwise direction. The field therefore decays in that direction. The

§ 5. Mathematical formulation

A mathematical formulation of the problem is as follows. We assume a steady-state, incompressible, and inviscid fluid. We assume that the magnetic field is frozen into the fluid, and that the fluid is incompressible. We assume that the magnetic field is not perturbed by the motion of the fluid, and that the fluid is not perturbed by the magnetic field. We assume that the fluid is infinite in extent and that the magnetic field is uniform in space.

Equation (9) and $\mu$ at the surface of the star are

and
lines of force in fig. 2 (a) can be regarded as being broken and rejoined to form those shown in fig. 2 (b). The total length of the lines of force decreases in the process, and it follows that the energy of the field decreases. This is necessary, since the energy for the discharge must be supplied by the field, if the material is initially static. Figure 3 shows two simple examples. In (a) two parts of a loop of force are close together with their fields in opposite directions, and the result is that the loop of force breaks into two loops, whose total length is less than that of the original loop. In (b) the reverse process occurs, but the length of the final loop is less than the combined length of the original two loops. In both cases field energy is released and field energy from a relatively large region is concentrated on the particles in the neighbourhood of the neutral point.

Because the argument in this section is based on diagrams, we ought to consider the nature of the field in a plane parallel to the paper, but a short distance away from it. At a short enough distance the field is similar to that in the plane of the paper, but there is a small component perpendicular to this plane. The motion of the gas is also similar and $|\text{curl} \mathbf{H}|$ is very large in a small region near the neutral point. The field is not frozen into the gas in this region and the lines of force can be regarded as being broken and rejoined in the way just described. The discharge extends in the direction perpendicular to the paper up to a distance where the change in field is considerable.

§ 5. Mathematical Treatment of Neutral Points Neglecting the Pressure Gradient

A mathematical treatment of this instability is possible, if the pressure gradient is omitted, and provides an estimate of the time required for the discharge to start. The equations of motion are (3) and the hydrodynamical equation

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + (\text{curl} \mathbf{H}) \times \mathbf{H} / 4\pi \mu. \quad \ldots \ldots \quad (4)$$

We use a frame in which the neutral point is initially a stagnation point of the motion, and then the neutral point remains a neutral point and stagnation point throughout the motion.

Writing $u_{ij}$ and $H_{ij}$ for the tensors $\partial u_i / \partial x_j$ and $\partial H_i / \partial x_j$, and remembering that $u$ and $\mathbf{H}$ vanish at the neutral point, we obtain

$$\frac{\partial H_{ij}}{\partial t} = -u_{kj} H_{ik} + H_{kj} u_{ik} - H_{ij} u_{kk} \quad \ldots \ldots \quad (5)$$

and

$$\frac{\partial u_{ij}}{\partial t} = -u_{kj} u_{ik} + (H_{it} - H_{ti}) H_{ij} / 4\pi \mu. \quad \ldots \ldots \quad (6)$$

Also

$$\frac{\partial \mu}{\partial t} = -\mu u_{kk}. \quad \ldots \ldots \ldots \ldots \quad (7)$$

Equations (5), (6) and (7) determine the time derivatives of $u_{ij}$, $H_{ij}$ and $\mu$ at the neutral point in terms of these variables themselves. If the pressure gradient were included in eqn. (4) higher derivatives of the velocity would be involved and the number of variables would be infinite.
However, it is useful to study these equations and rely on physical arguments to discuss the effect of the pressure gradient, which depends on the state of the rest of the system. Also, the equations corresponding to (5) and (6) at any point other than a neutral point involve higher derivatives of the velocity and magnetic field, so that the mathematical method used here breaks down, and it is necessary to fall back on the physical argument, which has been given in § 3.

We take the case in which all components of both $H_{ij}$ and $u_{ij}$ with either suffix equal to 3, except $H_{33}$, vanish initially; then they vanish throughout. Typical equations for the other components are

\[
\begin{align*}
\partial H_{11}/\partial t &= u_{12}H_{21} - u_{21}H_{12} - H_{11}(u_{11} + u_{22}), \\
\partial H_{12}/\partial t &= u_{12}(H_{22} - H_{11}) - 2u_{22}H_{12}, \\
\partial u_{11}/\partial t &= -u_{11}^2 - u_{12}u_{21} + (H_{12} - H_{21})H_{21}/4\pi\mu, \\
\partial u_{12}/\partial t &= -u_{12}(u_{11} + u_{22}) + (H_{13} - H_{21})H_{22}/4\pi\mu. 
\end{align*}
\]  

Consider the state in which the current density vanishes; let the axes be chosen so that $H_{12}$ and $H_{21}$ vanish, and let $H_{11}$ be positive and $H_{22}$ negative. Also let all components of $u_{ij}$ vanish, and consider a perturbation in $H_{12}$, $H_{21}$, $u_{12}$ or $u_{21}$. Remembering that $\mu$ is always positive, the eqns. (8) show that the signs of the components of $H_{ij}$ and $u_{ij}$ will at first be given by one of the schemes in table 1.

<table>
<thead>
<tr>
<th>$H_{11}$</th>
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<th>$H_{21}$</th>
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It can then be seen that every term in the derivative of each component has the same sign as that component. Consequently all the components grow in magnitude and, since $H_{13}$ and $H_{23}$ have opposite signs, the current density grows. Until $H_{13}$, $H_{21}$, $(4\pi\mu)^{1/2}u_{12}$ and $(4\pi\mu)^{1/2}u_{21}$ are comparable with $H_{11}$ and $H_{22}$, they grow exponentially with a time constant $(4\pi\mu)^{1/2}|H_{11} - H_{22}|$. An example has been computed on the EDSAC in the Mathematical Laboratory, Cambridge, by S. Gill. After the components $H_{13}$, $H_{21}$, $(4\pi\mu)^{1/2}u_{12}$ and $(4\pi\mu)^{1/2}u_{21}$ become comparable with $H_{11}$ and $H_{22}$, all the components become infinite in a time of the order of the initial value of $(4\pi\mu)^{1/2}/H_{11}$, as is to be expected from the quadratic form of (8); $\mu$ does not increase appreciably until the later stage. We conclude that, if the pressure gradient does not oppose the motion, the situation is unstable in such a way that a discharge can be started by a small perturbation, and that the time required for the current density to grow is not many times larger than the initial value of $(4\pi\mu)^{1/2}/H_{11}$.

§ 6. TWO-DIMENSIONAL MODELS WITH NEUTRAL POINTS

For the purpose of obtaining a simple illustrative model one obvious simplification is to make the field two-dimensional, by taking $H_z = 0$, $\partial H_x/\partial z = \partial H_y/\partial z = 0$. Consider the configuration in static equilibrium.
In the simplest case the lines of force are concentric circles but there is then no X-type neutral point. If there is an X-type neutral point there is a line of force shaped like a figure 8 as shown in fig. 4 (a), where there is zero current density at the X-type neutral point. The shape of the lines of force in equilibrium depends on the relative strength of the field in different regions and can be discussed, using the fact that the energy in any thin tube of force increases with the length of the tube.

If the field in any particular tube were much stronger than in any of the others, this tube would take up a nearly circular shape, just as in the simplest case all the tubes are circular. If the magnetic energy inside the loops greatly exceeds that outside, the lines of force inside will approximate to concentric circles and in the extreme case the figure 8 will consist of two circles in contact as in fig. 4 (b). If the magnetic energy outside the figure 8 is much the greater, the lines of force outside will approximate to circles and in the extreme case the figure 8 will consist of two D’s back to back as in fig. 4 (c). In any intermediate case the configuration will be intermediate between figs. 4 (b) and (c), as shown in fig. 5. In each of these cases the angle between the limiting lines of force at the neutral point is zero. Also the field is symmetrical about the line through the three neutral points, which we now take as x-axis, the origin O being taken at the X-type neutral point. Consider a line of force inside one of the loops of the figure 8 and cutting the x-axis in P and Q, and let its curvature at P
be $K_P$ and at $Q, K_Q$. Obviously for fig. 4 (b) $|K_P| = |K_Q|$ and for fig. 4 (c) $|K_P| > |K_Q|$ so that in any case

$$|K_P| \geq |K_Q|.$$  \hspace{1cm} (9)

This result can be used to prove that $|\text{curl } \mathbf{H}|$ is infinite at $O$ if $|\mathbf{H}|$ is finite at $A$ and $B$.

The equation for static equilibrium is $\mathbf{j} \wedge \mathbf{H} / c = \nabla \mathbf{p}$ which yields $\text{curl} (\mathbf{j} \wedge \mathbf{H}) = 0$ and for two dimensional fields this reduces to

$$\mathbf{(H} \cdot \nabla) \mathbf{j} = 0.$$  \hspace{1cm} (10)

Equation (10) shows that $\mathbf{j}$ is constant on a line of force, but (10) is automatically satisfied at a neutral point, so that $\mathbf{j}$ can have any value at a neutral point. The vector potential $\mathbf{A}$ for the magnetic field can be taken as $(0, 0, A)$ and then $\mathbf{j}$ is $(0, 0, -c \nabla^2 A / 4\pi)$. On a line of force $A$ is constant and (10) shows that $\nabla^2 A$ is also constant, so that at a pair of points situated like $P$ and $Q$,

$$A_P = A_Q, \quad \nabla^2 (A)_P = \nabla^2 (A)_Q.$$  \hspace{1cm} (11)

Consider the variation of $A$ on the $x$-axis, so that $\partial^2 A / \partial x^2$ can be written

$$\frac{dA}{dx} \frac{dA}{dx},$$

and $\partial^2 A / \partial y^2 = K dA / dx$ where $K$ is the curvature of the line of force defined to be positive when the line of force is convex towards the direction of positive $x$.

Combining these results

$$\frac{1}{2} \frac{d}{dA} \left[ \left( \frac{dA}{dx} \right)^2 \right] + K \frac{dA}{dx} = \nabla^2 A.$$  \hspace{1cm} (12)
The points P and Q move along the x-axis as A varies and (11) and (12) yield
\[
\frac{d}{dA} \left[ \left( \frac{dA}{dx} \right)_Q^2 - \left( \frac{dA}{dx} \right)_P^2 \right] = 2 \left( K_P \frac{dA}{dx}_P - K_Q \frac{dA}{dx}_Q \right).
\]  \hspace{1cm} (13)

Now \( K_P < 0 \) and \( K_Q > 0 \). Suppose that P and Q move apart as A increases so that \( (dA/dx)_P < 0 \) and \( (dA/dx)_Q > 0 \). Then the right-hand side of (13) \( \geq 0 \) as
\[
|dA/dx|_Q \leq |K_P/K_Q||dA/dx|_P.
\]

Since \( (dA/dx)_P \) and \( (dA/dx)_Q \) tend to zero as P and Q approach R, and remembering (9), we conclude that
\[
|dA/dx|_Q > |dA/dx|_P \text{ or } |H_Q| > |H_P|.
\]

The above argument is valid so long as PQ lies inside AO and hence if \( |H| \) is not zero in the neighbourhood of A, \( |H| \) is finite at any point \((-\epsilon, 0)\) where \( \epsilon \) is small but not zero. Similarly if \( |H| \) is not zero in the neighbourhood of B, \( |H| \) at \((+\epsilon, 0)\) is finite. The field is directed in opposite directions at \((-\epsilon, 0)\) and \((+\epsilon, 0)\), hence at O the field must change discontinuously and \( \text{curl} \ H \) must be infinite.

The situation in which there is an infinite current density may be regarded as the extreme case of constriction. Constriction is usually discussed in connection with a field whose lines of force are concentric circles, and then the constriction is usually limited by the gas pressure. The difference between this case and that of an X-type neutral point is that in the latter the material can escape from the region of high current without crossing lines of force.

\section{7. Orbits of the Particles}

The foregoing arguments for particular conditions show that the current density at a neutral point increases so long as eqn. (3) is a valid approximation, and we are justified in stating that the current density becomes very large. A thorough discussion of the behaviour of such a discharge when our approximation breaks down will not be given here, as the calculation of the current due to the accelerated particles, when they have left the accelerating region, is too difficult and depends on the configuration of the field outside the accelerating region. A rough discussion is attempted in order to obtain an estimate of the importance of these discharges.

It has been seen that the accelerating region must be exceedingly thin in one direction, which is clearly the horizontal direction in fig. 1. It may extend to any distance in the vertical direction and so can be regarded as a very thin sheet. The effect of the magnetic field on the orbit of a particle during acceleration is important. Consider an orbit passing through the neutral point in a direction almost perpendicular to the plane of fig. 1. If it deviates in the horizontal direction it is brought back by the magnetic field, so that the orbit stays in the accelerating region even though this is very thin. If it deviates in the vertical direction, the magnetic field bends
it further away, so that the orbits fan out in the vertical direction. After leaving the accelerating region the particles can be regarded as moving along lines of force, if these are regarded as moving with the material. Now in the plane containing the perpendicular to the plane of fig. 1 and one of the other principle axes, the lines of force all pass through the neutral point. Consequently an orbit approximately follows one of these lines of force, and since the orbits in the accelerating region fan out, they will continue to spread after leaving the accelerating region, and the current density due to the accelerated particles will be much smaller than it is in the accelerating region. Outside the accelerating region there is a background of unaccelerated particles, and, since the accelerated particles will be considerably less numerous than these, they will probably neutralize the space charge and current density of the accelerated particles. We therefore suppose that a steady state is set up as described in § 4.

§ 8. Applications

It is now desirable to obtain a rough estimate of the voltage driving a discharge at a neutral point. Suppose that a steady state is set up as described in § 4. The electric field can be considered as the sum of the part driving the discharge and the induced field \(-u \times H/c\). Then since curl \(E\) must vanish in a steady state, it can be concluded that the electric field driving the discharge is of the same order as the induced field outside the discharge. In fig. 1 the induced field is everywhere directed into the paper and so also is the field required to drive the discharge. If the particles are accelerated over a distance \(l\), they will acquire energies of order \(e|u||H|/c\). In the following it will be assumed that the discharge extends in the direction of the current over a distance of order \(a\), the characteristic length of the system. It is found that collisions of the accelerated particles are not important in the applications discussed and then \(l \sim a\). Particles with momentum less than \(e|H|a/c\) move in orbits which spiral round the lines of force. For particles with relativistic energy the corresponding energy is approximately \(e|H|a\), so that particles accelerated at a neutral point do not acquire sufficient momentum to escape across the magnetic field. The energies involved are nevertheless very large; rough values for particular applications will now be briefly discussed.

The values of the relevant quantities are most accurately known for Hoyle's suggested theory of the aurora (Hoyle 1949). According to this a beam of ionized gas with a magnetic field frozen into it is emitted by the sun, neutral points occur in the neighbourhood of the earth, and the aurora is due to particles accelerated at these neutral points, which then travel along lines of force until they penetrate the atmosphere of the earth. The motion of the beam sets up currents at the neutral points which flow in a particular direction; the pressure gradients set up by the motion of the beam then reinforce the electromagnetic forces near the neutral point, and the result obtained in § 5 shows that discharges will occur. When a steady state is set up at the neutral point, the field in the earth's atmosphere is subject to a steady deflection where \(d\) and even \(l\) are.

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steady state is set up, the neutral points are stationary relative to the earth. The velocity of the beam is inferred from the delay between the observation of a solar flare and the commencement of a magnetic storm and is about $10^3$ cm/sec. Hoyle estimates the strength of the magnetic field in the beam at about $10^{-3}$ gauss, so that the electric field is of order $10^{-3}$ volts/cm. The only collisions that could be important are collisions with charged particles (i.e. encounters in which the particle suffers a large deflection) and the mean free path for such collisions is of order $w^2/ne^4$, where $w$ is the energy of the particle. $n$ is probably about 100 particles/cm$^3$ and even if $w$ is only the thermal energy, say $10^{-14}$ ergs., the mean free path is of order $2 \times 10^7$ cm. These collisions can therefore be neglected, because the mean free path increases as the energy of the particle increases. Hoyle supposes that the particles are accelerated over a distance $4 \times 10^7$ cm. This may be an underestimate, but it is sufficient to obtain particles of energy $4 \times 10^4$ ev, and this is the energy required for the particles to penetrate to a height of 100 km above the surface of the earth.

Giovanelli (1947, 1948) first suggested the possibility of discharges occurring at neutral points in connection with solar flares, which occur in the neighbourhood of large sunspots. He discusses collisions with neutral atoms and finds that these will be unimportant, if $|E|$ exceeds $E_L$, where $E_L$ depends on the height in the chromosphere, and has its maximum value, about $10^{-3}$ e.s.u., at the base of the chromosphere. If $|H| \sim 1000$ gauss, this would require only $|u| \sim 3 \times 10^4$ cm/sec so that collisions can again be neglected. If $4\pi \mu |u|^2 \sim |H|^2$, and $\mu \sim 10^{-13}$ g/cm$^3$, $|u|$ must be of order $10^6$ cm/sec. (This estimate is made purely on theoretical grounds and no such large velocities have been observed.) Then taking $a \sim 10^9$ cm, particles will be accelerated to $10^{13}$ ev. Even if this estimate is a factor of 1000 too high, soft cosmic rays would be produced and they could account for the large increases sometimes observed in the total intensity of cosmic rays at the time of intense solar flares (Forbush 1946, Neher and Roesch 1948).

We may also speculate on the possibility of discharges at neutral points in interstellar space using the values given by Fermi (1949). He describes a process by which particles could acquire energy over a very long time, and which shows promise of explaining the spectrum of cosmic rays. He discusses the collision processes which occur and they can certainly be neglected in a discharge. He gives the values $|H| \sim 10^{-5}$ gauss, $|u| \sim 3 \times 10^6$ cm/sec and $a \sim 10^{19}$ cm, which lead to an energy of $3 \times 10^{12}$ ev. These values are very uncertain; Batchelor (1950) believes the value of the magnetic field strength to be considerably too large. It may also be noted that Fermi requires heavy positive ions to be accelerated to about $10^6$ ev before his acceleration process will work, and that this could be achieved in solar flares and of course in other stars.

Electrical discharges are probably important as sources of radio noise. Outbursts of radio noise are known to be associated with solar flares, and it is possible that radio stars are also associated with discharges.
On the Conditions for the Occurrence of Electrical Discharges

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REFERENCES


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