Spontaneous current sheets and break-up of magnetic flux surfaces

R. Bhattacharyya[1,2], B. C. Low[2] and P. Smolarkiewicz[3]

National Center for Atmospheric Research, P. O. Box 3000, Boulder, CO 80307

[1] Advanced Study Program, [2] High Altitude Observatory

[3] Microscale & Macroscale Meteorological Division and Institute of Mathematics for Geosciences

We demonstrate spontaneous current sheet formation during the relaxation of a three dimensional magnetic field in a viscous, perfectly conducting incompressible magnetofluid. The current sheet manifests itself in the form of magnetic tangential discontinuity created when different parts of the fluid press each other as it relaxes to the lowest magnetic energy state. One novel feature of the numerical scheme used for this purpose is the description of the magnetic field in terms of evolving flux surfaces which are possible sites of tangential discontinuity formation. The computation follows initial global flux surfaces of simple geometry as they evolve in time to more complex forms creating magnetic tangential discontinuities in the process. This work illustrates the physics of spontaneous current sheet formation as described in the Parker theory.

The Physical Problem

Let us consider an incompressible viscous magnetofluid under the condition of perfect electrical conductivity initially at in-equilibrium. For simplicity, let us further assume the field lines to be untwisted so that they can be represented by two independent family of global flux surfaces.

- Infinite conductivity guarantees invariance of magnetic topology as the fluid evolves with time in response to the unbalanced Lorentz force. Magnetic energy gets converted into kinetic energy of flow and is lost irrecoverably through viscous dissipation

- Magnetic field being frozen into the plasma (infinite conductivity) can not vanish entirely and the system will reach a minimum magnetic energy equilibrium state with time.

- The prerequisite condition for equilibrium is that a set of flux surfaces are also isobaric surfaces. But from early writings of Grad [1] to the Parker theory of spontaneous current sheet formation [2], it is well known that the organization of these particular global flux surfaces into isobaric surfaces is generally not possible for three-dimensional magnetic field. Yet the viscous flow must terminate the field evolution as the field runs out of free energy under the frozen-in condition.

- The minimum energy equilibrium state, in general, is not continuous in space. As the fluid approaches its terminal state extreme gradients in fluid displacement and tangential discontinuities in magnetic field or current sheets form unavoidably.

- This work presents the first and simplified study in a series of investigations to understand the dynamical evolution of the plasma as it progresses toward the formation of current sheets under frozen-in condition

Governing Equations

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{j}\times\mathbf{B} + \mu_0\nabla^2\mathbf{v}$$ ⟶ Force balance equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})$$ ⟶ Induction equation

$$\nabla\bullet\mathbf{B} = 0$$

Simplification

$$\nabla\bullet\mathbf{v} = 0$$ ⟶ Incompressible flow

To avoid unnecessary complications due to internal fluid energy

- The central concept behind current sheet formation

As two fluid elements pertaining to two different portions of the same flux tube or two different flux tubes with their individual embedded magnetic field press into each other under frozen-in condition, the preservation of topology prohibits them from intermixing. Tangential discontinuities thus develop at the surface separating the interacting flux surfaces.

For visualization of the physical process we describe the magnetic field in terms of magnetic flux surfaces

Simplification ⇒ Untwisted magnetic field

$\mathbf{B} = \nabla\xi\times\nabla\zeta$ ξ, ζ are Euler potentials [3] ξ =constant, ζ=constant; represents a pair of flux surfaces

Time evolution equation for these surfaces are obtained from the induction equation

$$\frac{\partial \xi}{\partial t} + (\mathbf{v}\bullet\nabla\xi) = 0$$
$$\frac{\partial \zeta}{\partial t} + (\mathbf{v}\bullet\nabla\zeta) = 0$$

Along with

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{j}\times\mathbf{B} + \mu_0\nabla^2\mathbf{v}$$
$$\nabla^2 p = \nabla\bullet(\mathbf{j}\times\mathbf{B})$$ From incompressibility condition

$\mu_0 = \text{constant}$ $\rho = \text{constant}$

Constitutes the closed set of equations

Initial and boundary conditions

$$\xi(\mathbf{x},0) = \cos^2 z\ \sin y \qquad \zeta(\mathbf{x},0) = \cos^2 x\ \sin y$$
$$p(\mathbf{x},0) = 0 \qquad \mathbf{v}(\mathbf{x},0) = 0$$

- Triply periodic boundary condition to avoid complications of boundary walls
- Cartesian geometry employed

Computational Model

- A variant of the multi-scale computational fluid model EULAG [4] customized for the given problem is employed

The governing equations are solved numerically using a non-oscillatory forward-in-time (NFT) approach [5]

From the perspective of numerical approximation, fluid equations can be represented in the following prognostic form

$$\frac{\partial \rho\psi}{\partial t} + \nabla\bullet(\tilde{\mathbf{v}}\,\psi) = \rho\,\mathrm{R}$$ Ψ ≡ One of the three components of velocity/ Euler-potentials

R ≡ RHS including forcing terms and dissipative terms

$$\nabla\bullet \equiv \frac{\partial}{\partial \mathbf{x}} \qquad \tilde{\mathbf{v}} = \rho\dot{\mathbf{x}}$$ ⇒ Velocity

An EULAG's template algorithm for integrating the above equation over the temporal increment Δt can be symbolically written as

$$\psi_i^{n+1} = \frac{\rho_i^n}{\rho_i^{n+1}} A\left(\psi_i^n + 0.5\Delta t R_i^n, \tilde{\mathbf{v}}^{n+\frac{1}{2}}, \rho\right) + 0.5\Delta t R_i^{n+1}$$

$\psi_i^{n+1} \equiv$ Solution sought at grid point (t^{n+1}, x_i)

A denotes a second-order-accurate flux form forward truncation scheme, namely MPDATA (Multidimensional Positive Definite Advection Transport Algorithm), for integrating the homogeneous transport equation [5]

$$\frac{\partial \rho\psi_h}{\partial t} + \nabla\bullet(\tilde{\mathbf{v}}\,\psi_h) = 0$$

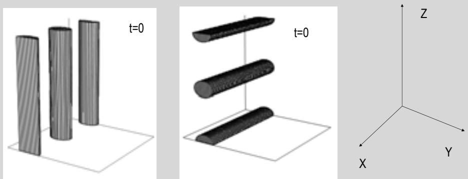
The template algorithm represents a system implicit with respect to pressure and all velocity components, because all the forcing terms are assumed to be unknown at n+1. It is solved by employing the mass-continuity equation

$$\nabla\bullet\tilde{\mathbf{v}} = 0$$

Our problem is greatly simplified due to the Euler potential representation of the magnetic field. The Lorentz force is known at n+1 and with constant density, mass continuity equation reduces to incompressibility condition.

Results and Discussions

- The initial Euler surfaces are periodic cylinders with non-circular cross-sections (Figure 1)
- Magnetic field lines on a given ζ = constant surface is defined by its intersection with constant ξ surfaces of different radii and vice versa
- All the field lines are closed with O-points lying along the cylindrical axis. Also for a given cylinder, there are three X-points one at each cap and one at the middle (Figure 2)



ζ= constant ξ = constant

Figure 1

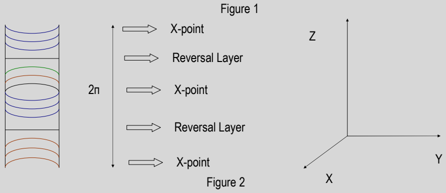


Figure 2

- The magnetic field reverses its sign at the quarter length of each cylinder

Our plan of action is to look for current sheets by examining different cross sections of these flux surfaces as they undulate in time

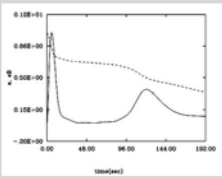


Figure 3. Time evolution global magnetic (dotted line) and kinetic energy (continuous line).

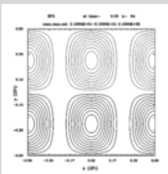


Figure 4. Temporal evolution of the CS of ζ= constant cylinder containing the X-point.

Results and Discussions (Contd.)

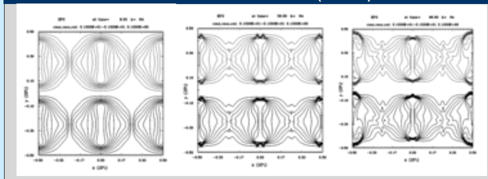


Figure 4 (continuation)

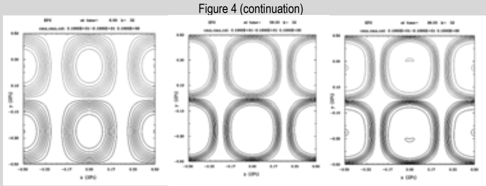


Figure 5. Temporal evolution of the reversal layer . Initial state is same as that in Figure 4.

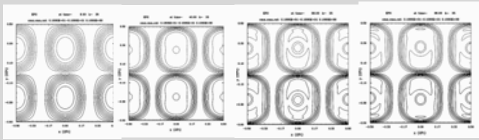


Figure 6. Temporal evolution of a layer in-between the reversal layer and X-point containing layer.

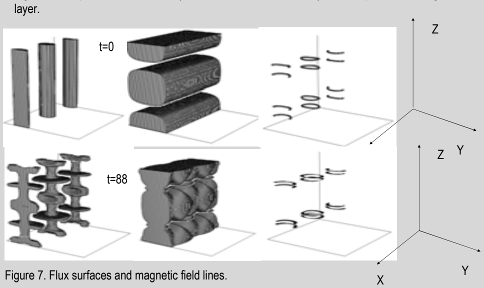


Figure 7. Flux surfaces and magnetic field lines.

- Numerical dissipation is predominantly effective from 96 time units (seconds) onwards (Figure 3)

- Initial Lorentz force stretches an outer cylinder outward (along x-axis) and an inner cylinder inward at the X-point containing layer (Figure 4).This generates plasma flow along the cylindrical axis and perpendicular to it (along y-direction). This is the initial surge of flow in Figure 3.

- As two field lines with opposite directions approach each other in the reversal layer, they mutually annihilate due to symmetry. The very existence of the O-pint on the cylinder axis guarantees that the magnetic field decreases radially inward. From confinement condition then, the kinetic pressure should be increasing radially inward, being maximum at the O-point.

- Local confinement at the reversal layer is lost and it bulges out. Kinetic pressure profile ensures that an inner cylinder bulges out more compared to an outer one (Figures 5 and 7)

- Plasma flow is incompressible and hence volume preserving. So an expansion perpendicular to the z-direction must be compensated by a corresponding contraction along the z-direction.

- These contractions will cause the magnetic field lines at a given side of the reversal layer to be more densely packed. Due to non-circular nature of the field lines magnetic pressure becomes asymmetric for a thin circular strip along the surface; being greater along the x-direction

- Eventual local necking and breaking of the flux surface (the magnetic field is steepened so much that numerical dissipation steps in locally and breaks the flux surface) points to the formation of current sheets in the process (Figures 6 and 7)

References

[1] H. Grad and H. Rubin,, *Proc. of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva) 31, 190 (1958).
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