Toward a Fully Consistent Radiation Hydrodynamics

presented to workshop

DimitriFest:
Recent Directions In Astrophysical Quantitative Spectroscopy and Radiation Hydrodynamics

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Dimitri Mihalas set the standard for all work in radiation hydrodynamics since 1984. The present contribution builds on *Foundations of Radiation Hydrodynamics* to explore the relativistic effects that have prevented having a consistent non-relativistic theory. Much of what I have to say is in FRH, but the 3-D development is new.
The conundrum in the title [The Radiation Transport Conundrum in Radiation Hydrodynamics] is whether to treat radiation in the lab frame or the comoving frame in a radiation-hydrodynamic problem.

Several of the difficulties are associated with combining a somewhat relativistic treatment of radiation with a non-relativistic treatment of hydrodynamics.

The principal problem is a tradeoff between easily obtaining the correct diffusion limit and describing free-streaming radiation with the correct wave speed.

The computational problems of the comoving-frame formulation in more than one dimension, and the difficulty of obtaining both exact conservation and full $u/c$ accuracy argue against this method.

As the interest in multi-D increases, as well as the power of computers, the lab-frame method is becoming more attractive.

The Monte Carlo method combines the advantages of both lab-frame and comoving-frame approaches, its only disadvantage being cost.
The summary was based on this scorecard

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Advantages</th>
<th>Disadvantages</th>
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<tr>
<td>pretend $\mathbf{u} = 0$</td>
<td>simplicity</td>
<td>radiation pressure and energy effects are lost entirely</td>
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<tr>
<td>mixed frame (lab-frame $\mathbf{u}$-expansion)</td>
<td>easy to solve the transport eq. (without scattering); exact conservation§</td>
<td>fails for problems with lines; difficult to treat scattering; complexity; dense mesh in $\nu$-$\mathbf{n}$</td>
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<tr>
<td>comoving frame moment eqs.</td>
<td>obtains diffusion limit; solve coupled radhydro problems with elliptic solvers; adapted to coupled RH Godunov method</td>
<td>frequency-dependent problem much more difficult to solve, esp. for non-monotone or multi-D flows; closures may be inaccurate; have to choose between conservation and full $\mathbf{u}/c$ accuracy</td>
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<tr>
<td>comoving-frame transport</td>
<td>obtains diffusion limit; no ad hoc closure</td>
<td>PDE difficult to solve for non-monotone or multi-D flows</td>
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<tr>
<td>Monte Carlo</td>
<td>exact§ apart from statistics</td>
<td>cost</td>
</tr>
<tr>
<td>lab-frame eqs. with exact (formal) sources</td>
<td>easy to solve; exact§ conservation</td>
<td>care is required with the sources and differencing to obtain diffusion limit; dense mesh in $\nu$-$\mathbf{n}$</td>
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§SR effects $O(\mathbf{u}/c)$ in hydro are neglected
The problem arises because the radiation field is described by an intensity function 
$I(\mathbf{r}, \mathbf{n}, \nu, t)$, and there is a choice of frame implicit in this function, one of

- Laboratory frame —at rest with respect to the system as a whole
- Comoving frame —obtained from the former by a Lorentz transformation with the local fluid velocity

The direction vector $\mathbf{n}$ and the frequency $\nu$ change in the transformation, as does the value of $I$.

The transport operator is simple in the lab frame and the material-coupling terms are simple in the comoving frame, so both choices have good and bad aspects.
The comoving frame is the frame of reference for $\nu$, $\mathbf{n}$ and $I_\nu$, nothing more. It says nothing about Eulerian vs. Lagrangean hydrodynamics, i.e., about the nature and motion of the computational mesh.

At one time I advanced a 1-D approach using Riemannian-geometry methods with a comoving coordinate system. This turns out not to be necessary in 1-D and impossible in 2-D or 3-D, except for irrotational flowws, a fatal limitation.

The most successful comoving-frame approach is to proceed from the lab frame transport equation by applying Lorentz transformations.
The exact transport equation in the lab frame is

\[
\frac{1}{c} \frac{\partial I_v}{\partial t} + \mathbf{n} \cdot \nabla I_v = j_v - k_v I_v
\]

in which \( j_v \) is the emissivity and \( k_v \) is the absorptivity, and both include scattering.

The effects of fluid motion are buried in \( j_v \) and \( k_v \).
The question arises whether or not to treat the kinematics relativistically in the radiation transport, when calculating the emission and absorption terms, or in the comoving-frame transport equation. The all-relativistic approach is consistent and recommended, but often we are called to couple radiation to non-relativistic hydrodynamics, which is the problem I want to consider. In this case there will be inconsistencies if $O(u^2/c^2)$ terms are retained in the kinematics.
Background—The Doppler-aberration transformations

\[
\begin{align*}
\nu &= \nu_0 \gamma_u \left(1 + \frac{n_0 \cdot u}{c}\right) \\
n &= \frac{\gamma_u u/c + n_0 + (\gamma_u - 1)(n_0 \cdot u)u/u^2}{\gamma_u(1 + n_0 \cdot u/c)}
\end{align*}
\]

where “\(0\)” quantities are in the comoving frame of the fluid, which moves with velocity \(u\), and \(\gamma_u = (1 - u^2/c^2)^{-1/2}\). Make \(u \ll c\) and get

\[
\begin{align*}
\nu &= \nu_0 \left(1 + \frac{n_0 \cdot u}{c}\right) \\
n &= \frac{n_0 + u/c}{1 + n_0 \cdot u/c}
\end{align*}
\]

Then \(I_\nu / \nu^3\) is a Lorentz invariant, so—

\[
I_\nu = \left(\frac{\nu}{\nu_0}\right)^3 I_\nu^0
\]
Background—The Euler equations with radiation coupling

\[
g^0 = \int dv \int_{4\pi} d\Omega \left(j_v - k_v I_v\right)
\]

\[
g = \frac{1}{c} \int dv \int_{4\pi} d\Omega \mathbf{n}(j_v - k_v I_v)
\]

\[
\frac{\partial}{\partial t} \left(\rho e + \frac{1}{2} \rho u^2\right) + \nabla \cdot \left(\rho u h + \frac{1}{2} \rho u u^2\right) = -g^0
\]

\[
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho uu) + \nabla p = -g
\]

Besides the usual symbols, \(h\) is the specific enthalpy of the material.

Important note: the radiation terms are all in the lab frame here!
Background—Comoving-frame coupling terms

\[
\begin{align*}
    g^0_0 &= \int d\nu_0 \int_{4\pi} d\Omega (j^0_v - k^0_v I^0_v) \\
    g_0 &= \frac{1}{c} \int d\nu_0 \int_{4\pi} d\Omega n_0 (j^0_v - k^0_v I^0_v)
\end{align*}
\]

from which it follows that

\[
\begin{align*}
    g^0 &= g^0_0 + u \cdot g_0 \\
    g &= g_0 + \frac{u}{c^2} g^0
\end{align*}
\]

to order \(u/c\). The second term in the equation for \(g\) is problematic. It is the same order as the momentum addition to the material caused by the increase of the relative mass density when the material gains energy, \(i.e.,\) a purely relativistic effect.
If we neglect the $u/c^2$ term in the $g$ equation, then the total material energy and momentum equations combine to yield the internal energy equation

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho u e) + p \nabla \cdot u = -g^0 + u \cdot g \approx -g_0^0$$

Notice: the *internal* energy equation contains the radiation coupling in the comoving frame, while the *total* energy equation has the coupling term in the lab frame. We have to keep the frames straight!
The comoving-frame approach

The comoving-frame method describes the radiation using $n_0$ and $v_0$, the direction vector and frequency as viewed by an observer comoving with the fluid. This is a particular case of using an arbitrary tetrad $\{e^\mu_a, a = 1, \ldots, 4\}$ as the basis for 4-momentum space at each point $\{x^\mu\}$ of spacetime, where the $e^\mu_a$ are any desired functions. Thus the 4-momentum components in the natural basis and in the tetrad basis are related by

$$p^\mu = e^\mu_a p^a$$

The functions $e^\mu_a$ form a $4 \times 4$ matrix of which the inverse is the matrix $e^a_\mu$. The crucial objects related to the $e^\mu_a$ are the Ricci rotation coefficients $\Omega^a_{bc}$ defined in the following way: Let a vector with tetrad components $M^a$ and natural components $M^\alpha = e^\alpha_a M^a$ be displaced parallel to itself along $dx^\alpha = e^\alpha_a dx^a$. Parallel displacement requires that $dM^\alpha = -\Gamma^\alpha_{\beta\gamma} M^\beta dx^\gamma$, in terms of the Christoffel coefficients $\Gamma$ of the basic manifold. But the gradient in the tetrad functions also produces a change in the tetrad components for the displaced vector. The result is

$$dM^a = -\Omega^a_{bc} M^b dx^c$$

with

$$\Omega^a_{bc} = e^a_\alpha e^\gamma_c e^\alpha_b \Gamma^\gamma_{\beta\gamma} = e^a_\alpha e^\gamma_c e^\alpha_b \Gamma^\gamma_{\beta\gamma} + e^a_\alpha e^\beta_c e^\gamma_b \Gamma^\gamma_{\alpha\beta\gamma}$$

in which the comma and semicolon signify ordinary and covariant differentiation.
The tetrad-component transport equation

We let \( I \propto I_v/v^3 \), \( a \propto v k_v \) and \( e \propto j_v/v^2 \) denote the invariant intensity, absorptivity and emissivity, respectively. Let \( s \) be an affine parameter on the photon’s null geodesic, so \( dx^\mu/ds = p^\mu \), where \( p^\mu \) is the 4-momentum. Then the invariant transport equation is

\[
\frac{dI}{ds} = e - a I
\]

The derivative on the left is evaluated using the result just found for \( dp^a \), with \( p^\mu = e^\mu_a p^a \)

\[
e^\mu_a p^a I, \mu - \Omega^a_{bc} p^b p^c \frac{\partial I}{\partial p^a} = e - a I
\]
NR — Results from the Ricci coefficients

In the $O(u/c)$ approximation the tetrad components of $p^b p^c \Omega^a_{bc}$ are found to be $\left( n_0 \cdot \frac{a}{c} + n_0 \cdot \nabla u \cdot n_0 + a + c n_0 \cdot \nabla u \right)$, and this leads to the transport equation in a form similar to Buchler’s (1983)

$$
\left( 1 + \frac{n_0 \cdot u}{c} \right) \frac{1}{c} \frac{\partial I^0}{\partial t} + \left( n_0 + \frac{u}{c} \right) \cdot \nabla I^0
$$

$$
- \frac{\nu_0}{c} \left( \frac{a}{c} + n_0 \cdot \nabla u \right) \cdot \nabla \nu_0 n_0 I^0 + \frac{3}{c} \left( \frac{n_0 \cdot a}{c} + n_0 \cdot \nabla u \cdot n_0 \right) I^0 = j^0 - k^0 I^0
$$
Different suggestions have been made about the ordering of the terms in the CMF equation. Letting the characteristic length scale and time scale be $L$ and $T$ shows that the terms in the transport operator have orders of $I^0/L$, $I^0/cT$, $uI^0/(Lc)$ and $uI^0/(c^2T)$.

- If $T$ is $O(L/c)$ (radiation fbw time scale) then the terms have order $I^0/L$ and $uI^0/(Lc)$, and all terms are needed for first-order accuracy in $u/c$.

- If $T$ is $O(L/u)$ (fluid-fbw time scale) then the orders are $I^0/L$, $uI^0/(Lc)$ and $u^2I^0/(c^2L)$, and the terms divided by $c^2$ are indeed second order in $u/c$ and can be dropped for a first-order solution.

- Both kinds of ordering have been, and are still, advocated by various authors.

- The conservative solution is to include all the terms, or, better yet, use the fully relativistic form.
Rather than considering $I^0$ as a function of the Cartesian tetrad components $\nu_0 n_0$, we can use spherical momentum coordinates: $\nu_0$ and the angles implicit in $n_0$. So instead of a momentum-space gradient term, we have a frequency-derivative term and an angle-derivative term:

$$
\frac{\nu_0}{c} \left( \frac{a}{c} + n_0 \cdot \nabla u \right) \cdot \nabla_{\nu_0 n_0} I^0 \rightarrow \\
\frac{1}{c} \left( \frac{n_0 \cdot a}{c} + n_0 \cdot \nabla u \cdot n_0 \right) \nu_0 \frac{\partial I^0}{\partial \nu_0} + \frac{1}{c} \left( \frac{a}{c} + n_0 \cdot \nabla u \right) \cdot (1 - n_0 n_0) \cdot \nabla_{n_0} I^0
$$

The former is the Doppler term and the latter is the aberration term. The factor $1 - n_0 n_0$ is the perpendicular projector relative to $n_0$ that converts $\nabla_{n_0}$ into the gradient on the unit sphere.
NR — Recovering lab-frame moment equations from CMF moments

Summing the CMF energy equation and the product of $\mathbf{u}$ with the momentum equation, and conversely, *then discarding the higher-order terms in $\mathbf{u}$*, leads to

$$\frac{\partial}{\partial t} \left( E_0 + \frac{2}{c^2} \mathbf{u} \cdot \mathbf{F}_0 \right) + \nabla \cdot \left( \mathbf{F}_0 + \mathbf{u} E_0 + \mathbf{u} \cdot \mathbf{P}_0 \right) = g_0^0 + \mathbf{u} \cdot \mathbf{g}_0$$

$$\frac{1}{c} \frac{\partial}{\partial t} \left( \mathbf{F}_0 + \mathbf{u} E_0 + \mathbf{u} \cdot \mathbf{P}_0 \right) + \nabla \cdot \left[ c \mathbf{P}_0 + \frac{1}{c} (\mathbf{u} \mathbf{F}_0 + \mathbf{F}_0 \mathbf{u}) \right] = c \mathbf{g}_0 + \frac{1}{c} \mathbf{u} g_0^0$$

which are equivalent to the lab-frame moment equations given earlier.

*Global energy and momentum conservation are obeyed only to $O(\mathbf{u}/c)$ when the comoving-frame equations of that order are used.*
NR — Making conservative CMF moment equations

Dropping the “small” terms from the CMF moment equations leads to this set:

\[
\frac{\partial E_0}{\partial t} + \nabla \cdot (uE_0 + F_0) + P_0 : \nabla u = \int d\nu (4\pi j^0_v - k^0_v cE_v^0) = g^0_0
\]

\[
c\nabla \cdot P_0 = -\int d\nu k^0_v F_v^0 = cg_0
\]

which satisfy this energy conservation law

\[
\frac{\partial E_0}{\partial t} + \nabla \cdot (F_0 + uE_0 + u \cdot P_0) = g^0_0 + u \cdot g_0
\]

This system does exactly conserve energy and momentum, at the cost of not being hyperbolic—with unbounded propagation speed, but the correct diffusion limit. The other problem is that these moment equations do not follow accurately from any form of the CMF transport equation.
For the limit \( \lambda \to 0 \), where \( \lambda \) is the radiation mean free path, there is an asymptotic relation for the comoving-frame intensity (the boxed terms are omitted in the simplified equation):

\[
I_{v_0}^0 \sim B_{v_0} - \lambda_{v_0} \frac{dB_{v_0}}{dT} \left( n_o \cdot \nabla T + \frac{1}{c} \frac{\partial T}{\partial t} + \frac{1}{c} \mathbf{u} \cdot \nabla T + \frac{n_o \cdot \mathbf{u}}{c^2} \frac{\partial T}{\partial t} + \frac{n_o \cdot \mathbf{a}}{c^2} T + \frac{n_o \cdot \nabla \mathbf{u} \cdot n_0}{c} T \right)
\]

with the implications

- The energy density tends to thermal equilibrium in the comoving frame
- The flux is \( O(\lambda) \) in the comoving frame
- Obtaining these results in a lab-frame calculation imposes constraints on the spatial differencing (see Mihalas and Auer [2001])
NR — The mixed-frame expansion method

Introduced by Fraser (1966), and developed by Hsieh and Spiegel (1976), Mihalas and Klein (1982), and Lowrie, Morel and Hittinger (1999), this method uses the lab-frame equations with $j_\nu$ and $k_\nu$ replaced by $O(u/c)$ expansions involving $j^0_\nu$ and $k^0_\nu$:

$$ j_\nu \approx j^0_\nu + \frac{1}{c} u \cdot n \left( 2j^0_\nu - \nu \frac{\partial j^0_\nu}{\partial \nu} \right) $$

$$ k_\nu \approx k^0_\nu - \frac{1}{c} u \cdot n \left( k^0_\nu + \nu \frac{\partial k^0_\nu}{\partial \nu} \right) $$

apart from scattering; coherent isotropic scattering in the fluid frame leads to a messy expression for $j_\nu$ involving frequency derivatives of the intensity. All these expansions fail when $u$ is comparable to or larger than a line width in velocity units—generally in any supersonic flow. For this reason this method is not used for problems involving lines.
Comment on the $O(u/c)$ approximations

When we combine the non-relativistic Euler equations with any variation of semi-relativistic radiation transport, we are nagged by inconsistencies that appear as lack of exact conservation. Or if certain terms are dropped in the radiation equations to allow exact overall conservation, then we are concerned about accuracy, and whether the radiation streams at the correct speed. Nor does using relativistically-exact transport alone solve the problem.
My suggestion —

Go Relativistic
Special Relativistic hydro equations

\[ \frac{\partial}{\partial t}(\rho \gamma_u) + \nabla \cdot (\rho \gamma_u \mathbf{u}) = 0 \]
\[ \frac{\partial}{\partial t} \left[ \rho c^2 \gamma_u (\gamma_u - 1) + \rho e \gamma_u^2 + p (\gamma_u^2 - 1) \right] \]
\[ + \nabla \cdot \left[ \rho c^2 \gamma_u (\gamma_u - 1) \mathbf{u} + \rho e \gamma_u^2 \mathbf{u} + p \gamma_u^2 \mathbf{u} \right] = c^2 f^0 \]
\[ \frac{\partial}{\partial t} \left[ \left( \rho + \frac{\rho e}{c^2} + \frac{p}{c^2} \right) \gamma_u^2 \mathbf{u} \right] + \nabla \cdot \left[ p \mathbf{I} + \left( \rho + \frac{\rho e}{c^2} + \frac{p}{c^2} \right) \gamma_u^2 \mathbf{u} \mathbf{u} \right] = \mathbf{f} \]

\( \gamma_u \) is the Lorentz factor corresponding to the fluid velocity \( \mathbf{u} \), and \( (f^0 \quad \mathbf{f}) \) is the (contravariant) 4-force in \( MT^{\mu \nu} = f^\mu \)
Internal energy and acceleration equations

\[
\rho \gamma_u \left[ \frac{D e}{D t} + p \frac{D}{D t} \left( \frac{1}{\rho} \right) \right] = c^2 g_{\mu \nu} U^\mu f^\nu
\]

\[
\rho \gamma_u \left( 1 + \frac{e}{c^2} + \frac{p}{\rho c^2} \right) \frac{D}{D t} (\gamma_u u) = -\nabla p + f - \frac{\gamma_u}{c^2} \left( c^2 g_{\mu \nu} U^\mu f^\nu + \gamma_u \frac{D p}{D t} \right) u
\]

The quantity \(c^2 g_{\mu \nu} U^\mu f^\nu\) is the projection of the 4-force on the 4-velocity; it is the energy source rate in the comoving frame, \(-g_0^0\)
Why is this easy?

Numerical methods for SRHD are quite a bit harder than for NR hydro, owing to the necessity for solving 5 nonlinear equations to get $\rho$, $u$ and $e$ from the conserved variables, and the complication of the Riemann solver due to the coupling of velocity components through the factor $\gamma_u$

But either

- $u/c$ is modest, $\gamma_u \approx 1$ and the SR equations are just slight perturbations of the NR ones, and similarly easy to solve, or

- you had to be doing SRHD anyway, right?
What is gained by using SRHD?

The radhydro equations are now exactly consistent. The lab-frame radiation moment equations are already in conservation-law form, and the source 4-vector $g^\mu$ precisely balances the one in the matter stress-energy conservation law. The relativistic relations between $g^\mu$ in the lab and fluid frames must be used—

$$g^0 = \gamma u \left( g^0_0 + u \cdot g_0 \right)$$

$$g = \left[ g_0 + \gamma u g^0_0 u + \frac{\gamma u - 1}{u^2} (g_0 \cdot u) u \right]$$
How do I ensure that the source terms are right?

Either use lab-frame radiation transport with Lorentz-transformed emissivity and absorptivity,

\[ \nu = \nu_0 \gamma_u \left( 1 + \frac{n_0 \cdot u}{c} \right) \]

\[ n = \frac{\gamma_u u/c + n_0 + (\gamma_u - 1)(n_0 \cdot u)u/u^2}{\gamma_u (1 + n_0 \cdot u/c)} \]

\[ j_\nu = j_{\nu_0}^0 \left( \frac{\nu}{\nu_0} \right)^2 \]

\[ k_\nu = k_{\nu_0}^0 \left( \frac{\nu_0}{\nu} \right) \]

or evaluate \( I^0 \) in the comoving frame using \( j_{\nu_0}^0 \) and \( k_{\nu_0}^0 \) and calculate \( g_0^0 \) and \( g_0 \) from that.
Oh, yes. What about the comoving-frame transport equation?

Recall that the definition of the CMF is that a special tetrad of 4-vectors is used for a basis of 4-momentum space, such that vector #0 is the fluid 4-velocity. In fact the tetrad \( e^\mu_a \) is just the Lorentz transformation matrix. The CMF equation is then

\[
e^\mu_a p^a \mathcal{I}, \mu - \Omega^a_{bc} p^b p^c \frac{\partial \mathcal{I}}{\partial p^a} = \varepsilon - a \mathcal{I}
\]

in which the \( p^a \) are the tetrad components of the 4-momentum, \((v_0 \; v_0 n_0)\), \(\mathcal{I}\) is the invariant intensity \(I^0/v_0^3\), \(\varepsilon\) is the invariant emissivity \(j^0_v/v_0^2\) and \(a\) is the invariant absorptivity \(v_0 k^0_v\). The \(\Omega\) coefficients are computed from

\[
\Omega^a_{bc} = e^a_{\alpha} e^\gamma_{c} e^\alpha_{b; \gamma} = e^a_{\alpha} e^\gamma_{c} e^\alpha_{b, \gamma} + e^a_{\alpha} e^\beta_{b} e^\gamma_{c} \Gamma^\alpha_{\beta \gamma}
\]
Oh, by the way, the CMF equation is in conservative form.

The same CMF equation can also be written

\[(e^\mu_a p^a \mathcal{I})_{;\mu} - \frac{\partial}{\partial p^a} \left( \Omega^a_{bc} p^b p^c \mathcal{I} \right) = e - a \mathcal{I} \]

The proof of this involves showing that covariant derivative of \(e\) balances an expression involving \(\Omega\).

If the covariant divergence is used in the first term, and if the \(\Omega\)s include the Christoffel coefficients, this equation is also good in curvilinear coordinates and in GR.
The CMF equation in terms of ordinary intensity

\[
\frac{dt}{ds} \frac{\partial I^0}{\partial t} + \frac{d\mathbf{r}}{ds} \cdot \nabla I^0 + \frac{d\mathbf{p}}{ds} \cdot \nabla_p I^0 - \frac{3}{v_0} \frac{d\nu_0}{ds} I^0 = j^0 - k^0 I^0
\]

\( dt/ ds \) and \( dr/ ds \) are the time and space components of \( p^\mu = e^\mu_a p^a \)

\[
\frac{dt}{ds} = \gamma u (1 + \mathbf{u} \cdot \mathbf{n}_0/c)/c = \frac{\nu}{\nu_0 c}
\]

\[
\frac{d\mathbf{r}}{ds} = \gamma u \mathbf{u}/c + \mathbf{n}_0 + (\gamma u - 1) \mathbf{u} \cdot \mathbf{n}_0 \mathbf{u}/u^2 = \mathbf{n} \frac{\nu}{\nu_0}
\]

Getting \( dp/ ds \) and \( d\nu_0/ ds \) is a bear; see the next slides
The CMF momentum-gradient coefficients

\[
\frac{dp}{ds} = -\frac{v_0}{c} \left\{ \frac{1}{c} \gamma_u^2 (a + u \cdot \nabla u) + \frac{\gamma_u^2 (\gamma_u - 1)}{cu^2} u \cdot (a + u \cdot \nabla u) u \\
+ \frac{\gamma_u (\gamma_u - 1)}{u^2} \left[ -(a + u \cdot \nabla u) \cdot n_0 u + (a + u \cdot \nabla u) u \cdot n_0 \right] \\
+ \frac{\gamma_u^2}{c^2} a (u \cdot n_0) + \frac{\gamma_u^2 (\gamma_u - 1)}{c^2 u^2} (u \cdot a) (u \cdot n_0) u + \gamma_u n_0 \cdot \nabla u \\
+ \frac{\gamma_u (\gamma_u - 1)}{u^2} \left[ (n_0 \cdot \nabla u \cdot u) u + (u \cdot \nabla u) (u \cdot n_0) \right] \\
+ \frac{\gamma_u (\gamma_u - 1)^2}{u^4} (u \cdot \nabla u \cdot u) (u \cdot n_0) u \\
+ \frac{\gamma_u - 1}{u^2} \left\{ -\gamma_u (n_0 \cdot a) (u \cdot n_0) / c + \gamma_u a (u \cdot n_0)^2 / c \\
- c u (n_0 \cdot \nabla u \cdot n_0) + c n_0 \cdot \nabla u (u \cdot n_0) \\
+ \gamma_u - 1 \left[ - c u (u \cdot \nabla u \cdot n_0) (u \cdot n_0) + c (u \cdot \nabla u) (u \cdot n_0)^2 \right] \right\} \right\}
\]
The CMF frequency-derivative coefficient

\[
\frac{dv_0}{ds} = -\frac{v_0}{c} \left\{ \frac{y_u^2}{c} (a + u \cdot \nabla u) \cdot n_0 + \frac{y_u^2(y_u - 1)}{cu^2} u \cdot (a + u \cdot \nabla u) (u \cdot n_0) \\
+ \frac{y_u^2}{c^2} (n_0 \cdot a) (u \cdot n_0) + \frac{y_u^2(y_u - 1)}{c^2 u^2} (u \cdot a) (u \cdot n_0)^2 + y_u n_0 \cdot \nabla u \cdot n_0 \\
+ \frac{y_u(y_u - 1)}{u^2} (n_0 \cdot \nabla u \cdot u) (u \cdot n_0) + \frac{y_u(y_u - 1)}{u^2} (u \cdot \nabla u \cdot n_0) (u \cdot n_0) \\
+ \frac{y_u(y_u - 1)^2}{u^4} (u \cdot \nabla u \cdot u) (u \cdot n_0)^2 \right\}
\]

There is a consistency check:

\[
n_0 \cdot \frac{dp}{ds} = \frac{dv_0}{ds}
\]

which is obeyed here
Advantages and disadvantages of the relativistic CMF equation

Same old advantages:

- easy to get the correct diffusion limit for the fluid-frame flux
- excessive numbers of frequencies and angles not needed for reasonable accuracy

Same old disadvantage:

- transport equation may be hard to solve stably and accurately in 3-D given a turbulent velocity field

Big new advantage of relativistic form:

- precise consistency of relativistic radiation moments with matter stress-energy conservation law and transport equation
Summary

• Several of the difficulties are associated with combining a somewhat relativistic treatment of radiation with a non-relativistic treatment of hydrodynamics — fixed with SRHD and relativistic transport

• The principal problem is a tradeoff between easily obtaining the correct diffusion limit and describing free-streaming radiation with the correct wave speed — fixed with SRHD and relativistic transport

• The computational problems of the comoving-frame formulation in more than one dimension, and the difficulty of obtaining both exact conservation and full \( u/c \) accuracy argue against this method — the first point is still an issue

• As the interest in multi-D increases, as well as the power of computers, the lab-frame method is becoming more attractive — more true every day

• The Monte Carlo method combines the advantages of both lab-frame and comoving-frame approaches, its only disadvantage being cost — this is still true; remaining inconsistencies gone with SRHD