THE ASTROPHYSICAL JOURNAL, 310:953–965, 1986 November 15

MODELS OF PARTIALLY OPEN MAGNETOSPHERES WITH AND WITHOUT MAGNETODISKS

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Received 1986 February 11; accepted 1986 April 28

ABSTRACT

This paper presents a large class of analytic solutions describing partially open magnetic fields in static equilibrium outside a central object, which may be taken to be a planet or a star. The problem for a potential magnetic field is first treated in axisymmetric geometry, with an equatorial, stress-free electric current sheet whose presence results in part of the magnetic flux opening to infinity. The solutions can be linearly superposed to construct idealized models of the solar coronal magnetic field in a partially open configuration. These solutions are further developed to allow for stresses in the current sheet and three-dimensionality, in that order of complexity. The stresses can be balanced in equilibrium by introducing gravitational and centrifugal forces acting on dense matter confined in the electric current sheet. Explicit solutions are presented to illustrate magnetic topologies of magnetospheres having rotating and nonrotating magnetodisks. A simple physical illustration is given to estimate the total mass in the Jovian magnetodisk from the observed macroscopic parameters of the disk electric current.

Subject headings: hydromagnetics — planets: Jupiter — planets: magnetospheres — stars: magnetic

I. INTRODUCTION

For a variety of astrophysical circumstances, inhomogeneous structures in a magnetic atmosphere may be approximated by electric current sheets. This approximation is useful if we are interested in the global configuration of the embedding atmosphere rather than the detailed internal structures of the inhomogeneities. The solar quiescent prominence, the heliospheric equatorial current sheet, and the magnetodisks in the Jovian magnetosphere and around some stars are some examples of objects that have been conveniently modeled in terms of electric current sheets (Kippenhahn and Schlüter 1957; Hundhausen 1977; Vasyliunas 1979, 1983; Aly 1986). Current sheet models are difficult to construct because they pose free boundary problems. In this paper, we present a class of problems whose solutions can be written in closed analytic form. These solutions can be linearly superposed to generate a rich variety of magnetic configurations, a convenience similar to that encountered with potential magnetic fields.

Consider the boundary value problem for a potential magnetic field in the infinite space outside a unit sphere. In this problem, we seek a potential magnetic field $\boldsymbol{B}$ in the infinite space $r > 1$, where

$$\nabla \times \boldsymbol{B} = 0 , \qquad (1)$$

subject to a prescribed normal field component at $r = 1$, and we demand that $\boldsymbol{B}$ vanishes at infinity. As is well known, this problem reduces to a boundary value problem of the Neumann type in classical potential theory. In the absence of magnetic monopoles, the prescribed normal field component $r = 1$ must have zero net magnetic flux, and the solution gives a magnetic field which has a closed geometry and vanishes as fast as $1/r^3$ at large r. In other words, the magnetic field far away is predominantly that of a dipole. Let us consider a modified version of the above problem. We allow electric current sheets to exist in the infinite region $r > 1$ and let the magnetic field be potential everywhere except on the current sheets. We further demand that the current sheets be stress-free and their presence results in a prescribed fraction of the magnetic flux opening to infinity. The classical Neumann problem has to be modified to demand that $\boldsymbol{B}$ vanishes at infinity like $1/r^2$. This asymptotic form follows from Maxwell's equation $\nabla \cdot \boldsymbol{B} = 0$ applied to an open radial field. The shape and extent of the electric current sheet needed to open up the prescribed amount of magnetic flux becomes a free surface to be determined together with the magnetic field. This is a difficult free boundary problem which arises naturally in the modeling of large-scale magnetic fields of the solar corona in terms of potential fields (e.g., Altschuler and Newkirk 1969; Schatten, Wilcox, and Ness 1969; Pneuman and Kopp 1971; Hundhausen 1977; Yeh and Pneuman 1977; Levine *et al.* 1977). The solar corona is in a state of expansion, with the magnetic field in the outer reaches forced into an open configuration (Parker 1963). Low in the corona, the expansion is negligible, and approximate static equilibrium obtains. Higher up, the corona expands in a hydromagnetic flow. A very crude model for the coronal magnetic field in such an environment is to take it to be potential. To allow for the effect of the solar wind without directly treating the hydromagnetic flow, we can prescribe a fraction of the magnetic flux to be radial and open in the presence of current sheets. In § II, we construct a general class of solutions for the case of an axisymmetric system with the current sheet located in the equator. In a further development given in § III, we allow for stresses in the current sheet and variations in three dimensions. These stresses may be balanced in equilibrium against gravitational and centrifugal forces acting on dense matter in the electric current sheet. We shall present a variety of explicit solutions illustrating idealized models of partially open stellar magnetospheres embedding equatorial disks (e.g., Ghosh and Lamb 1978, 1979; Mestel and Ray 1985; Gleeson and Axford 1976; Hill and Carbary 1978; Connerney, Acuña and Ness 1981). We conclude the paper in § IV with a discussion of the results.

[1] The National Center for Atmospheric Research is sponsored by the National Science Foundation.

II. PARTIALLY OPEN POTENTIAL MAGNETIC FIELDS

Let us introduce the potential Φ, where

$$\boldsymbol{B} = \nabla \Phi . \tag{2}$$

Equation (1) is satisfied automatically, and the divergence-free condition on $\boldsymbol{B}$ becomes the Laplace equation

$$\nabla^2 \Phi = 0 . \tag{3}$$

We use spherical coordinates r, θ, ϕ and seek solutions of equation (3) such that Φ satisfies the boundary condition

$$r = 1 ,$$

$$\frac{\partial \Phi}{\partial r} = F(\theta, \phi) , \tag{4}$$

and vanishes at infinity with the asymptotic form

$$r \to \infty ,$$

$$\Phi \to \frac{P_0(\theta, \phi)}{r} , \tag{5}$$

where the normal field distribution $F(\theta, \phi)$ is given and P_0 is piecewise constant over the domain $0 < \theta < \pi$, $0 < \phi < 2\pi$. The form of P_0 determines a prescribed amount of the magnetic flux opening to infinity in a given zone on the sphere at infinity. The distribution F is assumed to have zero net flux over the unit sphere. The use of boundary condition (5) instead of a simple statement that Φ vanishes at infinity requires the presence of electric current sheets in $r > 1$. To keep the problem simple, we demand that only stress-free electric current sheets are admissible. Such a magnetic field exerts no force on the plasma medium and is a special case of a force-free magnetic field. In § III, we relax this constraint to consider stressed current sheets and account for their interaction with the plasma medium. In the present consideration, the task is to solve for the potential Φ in equations (3)–(5), together with the self-consistent stress-free electric current sheets.

The above problem is simple to solve if the desired magnetic field is everywhere open in $r > 1$. In that case, the construction proceeds by first dividing the boundary at $r = 1$ into regions of positive and negative polarities according to the signs of the prescribed normal field $F(\theta, \phi)$. Take all the negative polarity regions and reverse the sign of F. Let us call these regions Σ_-. With the modified boundary condition at $r = 1$, now solve the Laplace equation for a smooth Φ which vanishes at infinity. This is readily done by classical techniques. Since the boundary $r = 1$ with the modified boundary condition has positive magnetic flux everywhere, the solution gives a magnetic field with all lines of force extending from the boundary $r = 1$ to infinity. Now reverse the directions of just those lines of force connected to the boundary $r = 1$ in the regions Σ_-. This leads to a magnetic field which is potential everywhere except at the surfaces made of magnetic lines of force in $r > 1$ which connect to the boundary $r = 1$ just along the boundaries of Σ_-. The reversals of the magnetic field at these magnetic surfaces are associated with sheet currents. The magnetic field does not penetrate these current sheets, and the method of construction ensures that the magnetic pressure is continuous across these current sheets. The current sheets are therefore stress-free. The magnetic field so constructed is then the solution we seek to equations (3), (4), and (5) for the special case of a fully open magnetic configuration. If the magnetic field is only partially open, the above construction fails and the problem is not tractable in general. In the following, we present a large class of particular solutions for the partially open magnetic configuration in a system which is symmetric about an axis and is also symmetric about the equatorial plane defined relative to its axis of symmetry. Such a system admits magnetic fields with the simplest form of the current sheet, namely, one which is flat and lying in the equator. Let us introduce some mathematical preliminaries before proceeding to obtain these solutions by direct construction.

As we deal with axisymmetric magnetic fields in the rest of this section, it is convenient to use the following representation of $\boldsymbol{B}$:

$$\boldsymbol{B} = \frac{1}{r \sin \theta} \left(\frac{1}{r} \frac{\partial A}{\partial \theta} \hat{r} - \frac{\partial A}{\partial r} \hat{\theta} \right) , \tag{6}$$

where A is a scalar function and we consider the case of a strictly poloidal field. The magnetic field $\boldsymbol{B}$ given by equation (6) is automatically divergence-free. The lines of force are given by curves of constant values of A, and we refer to A as the stream function. The latter property facilitates the plotting of lines of force for graphical presentation. Applying equation (1) for a potential magnetic field, we obtain

$$\frac{\partial^2 A}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial A}{\partial \theta} \right) = 0, \tag{7}$$

which is the equivalent of the potential equation (3); here we deal with the stream function A instead of the potential function Φ. The well-known spherical harmonic potential functions

$$w_n = r^{-n-1} P_n(\cos \theta) , \tag{8}$$

satisfying equation (3), give potential magnetic fields which are also generated by the corresponding stream-function solutions to equation (7):

$$W_n = \frac{1}{n} r^{-n} P_n^1 (\cos \theta) \tag{9}$$

(see, e.g., Batchelor 1970), where n is an integer and P_n and P_n^1 are standard Legendre polynomials and associated Legendre functions respectively.

Consider the oblate spheroidal coordinates ξ, η, and ϕ, where

$$x = r \sin \theta \cos \phi = a(1 + \xi^2)^{1/2}(1 - \eta^2)^{1/2} \cos \phi , \tag{10}$$

$$y = r \sin \theta \sin \phi = a(1 + \xi^2)^{1/2}(1 - \eta^2)^{1/2} \sin \phi , \tag{11}$$

$$z = r \cos \theta = a \xi \eta . \tag{12}$$

In equations (10), (11), and (12), we relate the spherical and spheroidal coordinates through the Cartesian coordinates x, y, and z, and the constant a is the radial distance of the common foci of the confocal elliptic and hyperbolic surfaces of revolution generated by the constant values of ξ and η respectively (Morse and Feshbach 1953). It is useful to bear in mind that these surfaces of revolution are generated by ξ and η in the ranges $0 < \xi < \infty$ and $-1 < \eta < 1$ (see Appendix A). In spheroidal coordinates, equation (7) for the stream function of a potential magnetic field takes the form

$$(1 + \xi^2) \frac{\partial^2 A}{\partial \xi^2} + (1 - \eta^2) \frac{\partial^2 A}{\partial \eta^2} = 0 , \tag{13}$$

with the following axisymmetric, separable, solutions

$$A_n = (1 + \xi^2)^{1/2}(1 - \eta^2)^{1/2} Q_n^1(i\xi) P_n^1(\eta) , \quad (14)$$

where P_n^1 and Q_n^1 are, respectively, associated Legendre functions of the first and second kinds with integer index n, and $i = (-1)^{1/2}$. Of special interest are the odd-n solutions describing fields which are symmetric about the equator. We henceforth take n to be odd unless stated otherwise. The solutions A_n are singular on the disk of radius a lying in the equator centered at the origin. This disk is just the degenerate elliptic surface of revolution on which $\xi = 0$. Electric currents flow in the azimuthal direction in this disk and give rise to the oblately shaped potential magnetic field outside the disk. The singularity distributed over a disk corresponds to the point singularities, located at the origin, of the more familiar spherical harmonic solutions W_n given by equation (9). In both cases, we may interpret the potential fields to be due to surface electric currents represented by the associated singularities.

In the axisymmetric system which is symmetric about the equator, one possibility for a partially open magnetic field outside a unit sphere is to have an equatorial electric current sheet that extends to infinity with an inner radius larger than unity. Let us identify the parameter a with this inner radius. A large class of solutions describing such a magnetic field can be constructed from the odd-n functions A_n as follows. We first invoke a well-known theorem which states that, in spherical coordinates, if $\Phi(r, \theta, \phi)$ is a solution of the Laplace equation, then so is the function

$$\Phi'(r, \theta, \phi) = \frac{b^2}{r} \Phi\left(\frac{b}{r}, \theta, \phi\right) , \quad (15)$$

where b is an arbitrary positive constant. This is the Kelvin transformation that generates one potential function from another by the inversion transformation with respect to the sphere of radius b centered at the origin; e.g., Kellogg (1929). In terms of the stream function for an axisymmetric magnetic field, the Kelvin transformation takes the form

$$A'(r, \theta) = rA\left(\frac{b^2}{r}, \theta\right) , \quad (16)$$

with the statement that if A satisfies equation (7), so does A'. Let us express ξ and η in terms of spherical coordinates,

$$\xi^2 = -\frac{1}{2}\left(1 - \frac{r^2}{a^2}\right) + \frac{1}{2}\left[\left(1 - \frac{r^2}{a^2}\right)^2 + \frac{4r^2}{a^2}\cos^2\theta\right]^{1/2} , \quad (17)$$

$$\eta^2 = -\frac{1}{2}\left(\frac{r^2}{a^2} - 1\right) + \frac{1}{2}\left[\left(\frac{r^2}{a^2} - 1\right)^2 + \frac{4r^2}{a^2}\cos^2\theta\right]^{1/2} , \quad (18)$$

and define the functions

$$u^2 = -\frac{1}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{1}{2}\left[\left(1 - \frac{a^2}{r^2}\right)^2 + \frac{4a^2}{r^2}\cos^2\theta\right]^{1/2} , \quad (19)$$

$$v^2 = -\frac{1}{2}\left(\frac{a^2}{r^2} - 1\right) + \frac{1}{2}\left[\left(\frac{a^2}{r^2} - 1\right)^2 + \frac{4a^2}{r^2}\cos^2\theta\right]^{1/2} , \quad (20)$$

which are obtained from ξ and η respectively, by inversion with respect to a sphere of radius a centered at the origin. As functions of the spherical coordinates, ξ, η, u, and v have singular properties at the equatorial plane $\theta = \pi/2$. We shall have occasion to use the first derivatives of these functions evaluated at the equator, and these derivatives are presented in Appendix A. Identifying the constant b with a and applying the Kelvin transformation to A_n, we obtain the following stream functions:

$$S_n = r(1 + u^2)^{1/2}(1 - v^2)^{1/2} Q_n^1(iu) P_n^1(v) . \quad (21)$$

The inversion transformation takes the disk electric current sheet, with radius a, of the stream function A_n into an infinite current sheet, associated with S_n, that lies in the equator with a circular hole of radius a centered at the origin. Figure 1 displays the magnetic lines of force for the simple dipolar case of A_1 and S_1. The equatorial current sheets associated with S_n are not stress-free, as is evident in the particular example in Figure 1 showing kinks in the lines of force at the current sheet. The normal component of the field at the current sheet and the azimuthal electric current give rise to a Lorentz force directed in the radial direction. To render it stress-free, the following two things need to be done.

We first remove the magnetic flux threading across the equatorial current sheet by introducing an additional magnetic flux due to suitable magnetic sources located at the origin. To that end, let us evaluate the normal field component along $r > a$ at the equator, obtaining

$$(B_\theta)_{r > a, \theta = \pi/2} = \frac{a^2}{r^3}\left[\frac{dQ_n(iu)}{du}\right]_{u=0} \times \left[\frac{dP_n(v)}{dv} - n(n+1)\frac{1}{v}P_n(v)\right]_{v=\sqrt{1-a^2/r^2}} . \quad (22)$$

To obtain equation (22), we made use of the properties of u and v given in Appendix A. For odd n, the case of interest here, the factor on the right side of equation (22) involving $P_n(v)$ is a finite series in v^2. It follows that the right-hand side of equation (22) can be expressed as a finite series in a/r. By matching coefficients of powers of a/r, a suitable linear combination of the stream functions W_n, given by equation (9), can be superposed upon S_n such that the net field has zero normal field component at the equator in the region $r > a$. Let us denote the desired linear combination by

$$T_n = \sum_{k \leq n} \alpha_k^n W_k(r, \theta) , \quad (23)$$

where α_k^n are the constant coefficients chosen to eliminate the normal field component contributed by S_n at the current sheet. The combined field $S_n + T_n$ is due to electric current sources located in the equatorial current sheet and the singularity of T_n at the origin. Direct calculation shows that the dipolar field S_1 requires only the dipole field W_1 to eliminate the normal field component at the equatorial current sheet. For $n > 1$, more than one term in the series in equation (23) are needed. Figure 2*a* shows the magnetic lines of force of the combined field $S_1 + T_1$. Note that all lines of force are closed. In fact, the combined field has an infinite field strength at the inner edge of the equatorial current sheet. This feature arises from the closed-field topology. We can think of the equatorial current sheet as a rigid perfect electrical conductor that "knifes" into a closed field from infinity. The infinite conductivity of the current sheet structure excludes the exterior magnetic field, compressing the latter to an infinite strength at its inner radius. Associated with this infinite field strength is an outward Lorentz force localized at the inner edge of the current sheet, repelling the intruding rigid body. We go to the next step of construction, which opens up part of the magnetic flux and

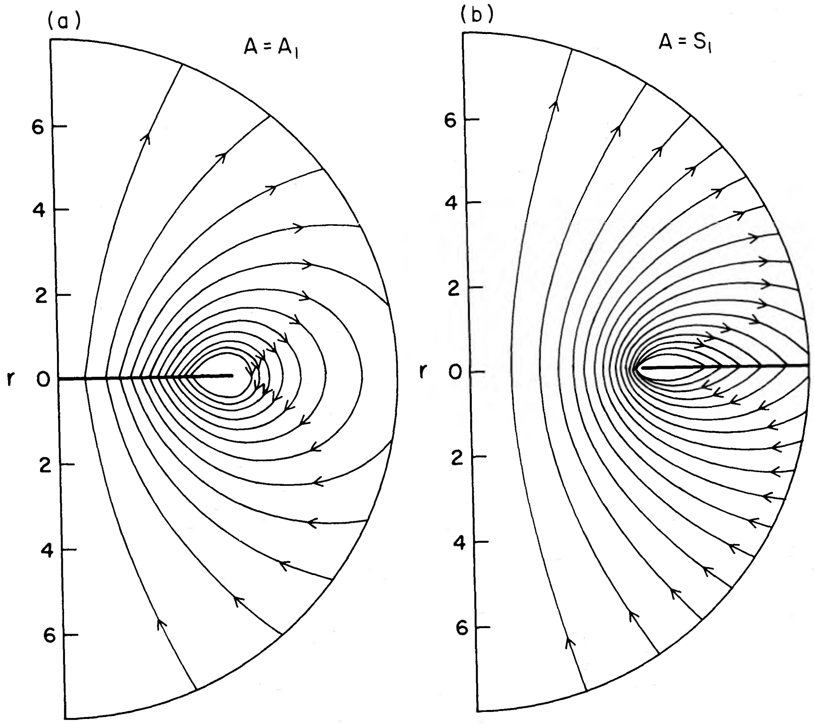

FIG. 1.—Lines of force in the (r, θ)-plane generated by the stream function A_1 and its Kelvin transform S_1 with $a = 4.0$. The thick lines here and in the other figures represent magnetic field sources in the form of electric current sheets or magnetic monopoles in the equatorial plane.

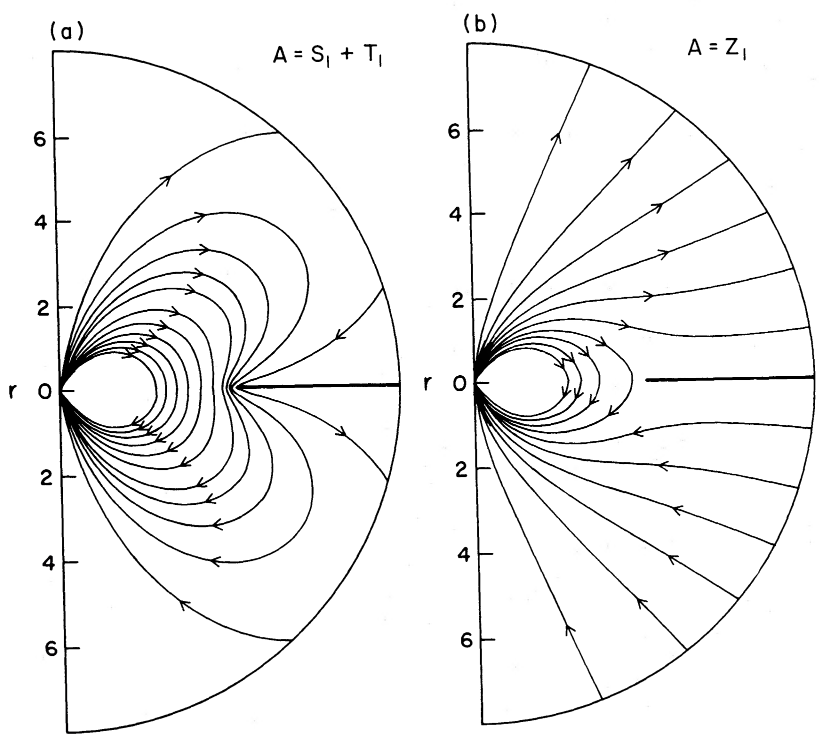

FIG. 2.—Lines of force in the (r, θ)-plane generated by the stream functions $S_1 + T_1$ and Z_1 with $a = 4.0$. (*a*) Current sheet stress-free except at the point $r = a$; (*b*) current sheet everywhere stress-free.

reduces the field strength at the inner edge of the current sheet to zero so as to remove the undesired Lorentz force.

Consider the special solution of equation (13)

$$A = \eta \, . \tag{24}$$

This stream function gives a potential magnetic field due to a disk of radius a centered at the origin, having a uniform distribution of magnetic monopoles. As shown in Figure 3*a*, the field lines are all normal to the disk source. If we reverse the direction of the lines of force on one side of the equator, as shown in Figure 3*b*, we would have a magnetic field that has no sources in $r < a$ but arises from the presence of an infinite equatorial current sheet with a circular hole of radius a centered at the origin. The stream function for this magnetic field is just

$$U = \pm \eta \, , \quad \text{if } \theta \gtrless \frac{\pi}{2} \, . \tag{25}$$

The singularity of the combined field $S_n + T_n$ at the inner edge of the current sheet arises solely from the field S_n, since T_n is singular only at the origin. It turns out that this singularity of S_n, for any n, is of the same order as a similar singularity of U at the same location. Thus, we can eliminate the singularity of S_n by surperposing S_n with U, introducing a suitable amplitude

$$\beta_n = -an(n+1)[Q_n(iu)]_{u=0} \left[\frac{dP_n(v)}{dv} \right]_{v=0} , \tag{26}$$

to arrive at the combined stream function

$$Z_n = S_n + T_n + \beta_n U \, . \tag{27}$$

This stream function describes a partially open potential magnetic field with a stress-free equatorial current sheet having an inner circular radius a. Figure 2*b* shows the lines of force for the case of $n = 1$.

Since potential fields may be superposed linearly, we have a large class of stream functions of the form

$$Z = \sum_n \gamma_n Z_n \, , \tag{28}$$

where γ_n are constant coefficients, each term Z_n is taken to have the same parameter a, and we remind the reader that n is restricted to odd integers. If we are given the normal component of an axisymmetric magnetic field at $r = 1$ as a boundary condition, we can use equation (6) to match the radial derivative of Z evaluated at $r = 1$ to determine the coefficients γ_n. To be consistent with the formulation of the problem, the boundary condition must correspond to a magnetic field which both is axisymmetric and has symmetry about the equator. Otherwise, the current sheet cannot be assumed to be lying in the equator. We are unable to prove the completeness of the set of functions Z_n in equation (28). Even if the set is complete, it remains a formidable task of inversion to obtain the coefficients γ_n because, unlike the classical case of spherical harmonics, the set of functions in equation (28) are not orthogonal. However, a rich variety of solutions can be generated from equation (28) by simply prescribing the constant coefficients γ_n. A simple example is shown in Figure 4. Figure 4*a* is a sketch of the lines of force for Z_3, showing the quadrupolar field morphology near $r = 1$. Note that the open part of the magnetic flux originates from mid-latitudes in the two hemispheres. The superposition

$$Z = 3a^2 Z_1 - 4Z_3 \tag{29}$$

gives the field shown in Figure 4*b*. Comparing this field with the field Z_1 shown in Figure 2*b*, we find that the contribution

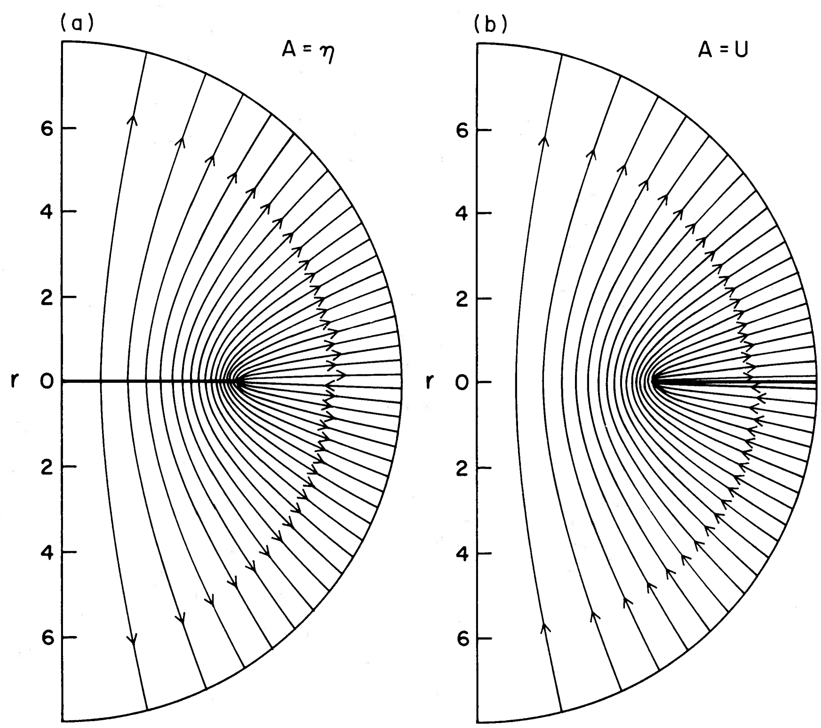

FIG. 3.—Lines of force in the (r, θ)-plane generated by the stream functions η and U with $a = 4.0$. (*a*) Field due to a disk, radius a, of uniformly distributed monopoles; (*b*) field due to an infinite equatorial current sheet.

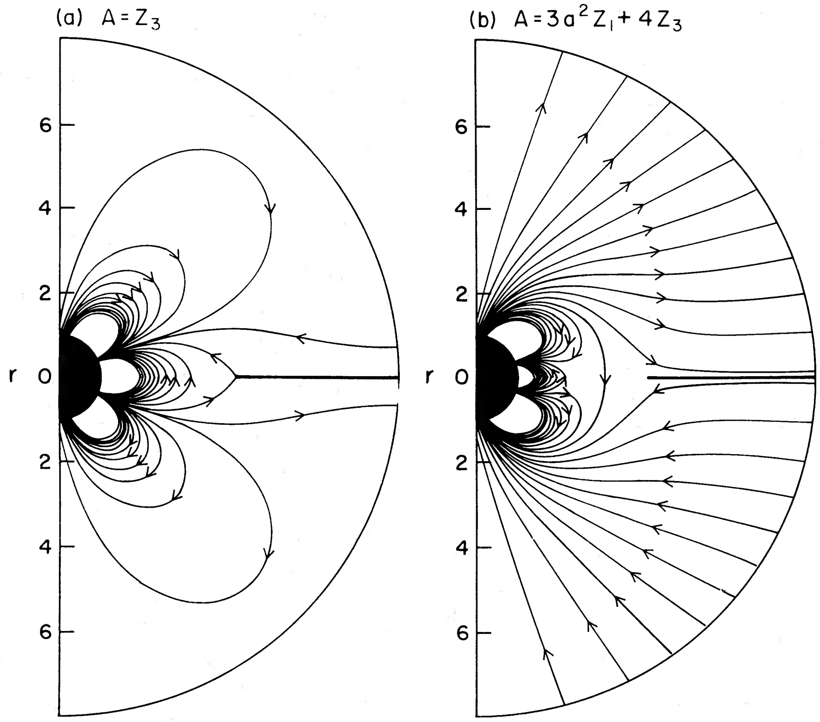

FIG. 4.—Lines of force in the $(r\ \theta)$-plane generated by the stream functions Z_3 and $3a^2Z_1 + 4Z_3$ with $a = 4.0$

of Z_3 to the former gives rise to the multipolar topology in the equatorial closed field region in Figure 4*b*. For the interested reader, Appendix B contains explicit functional forms of Z_1 and Z_3. We have chosen to work with stream functions for the convenience of using them to plot magnetic lines of force. A similar construction of magnetic fields using the potential function is outlined in Appendix B.

III. MAGNETOSPHERES WITH EQUATORIAL DISKS

The stream functions Z_n that generate Z in equation (28) all have stress-free current sheets of the same geometric form, namely, the infinite equatorial plane with a circular hole of radius a centered at the origin. Linear superposition gives the resultant stream function Z a stress-free current sheet of the same geometric form. Suppose the superposition is not limited to the set Z_n. Let us superpose Z, givey by equation (28), with an arbitrary potential stream function A which contains no current sheet in $r > 1$. The potential A will contribute an additional magnetic flux that threads across the equatorial current sheet, resulting in a stress in the current sheet. To have equilibrium, we must introduce body forces to balance this stress in the current sheet. In this section, we consider the case where the magnetic stress is balanced in equilibrium by gravitational and centrifugal forces acting on dense matter in the electric current sheets. A rich variety of magnetic configurations of interest to modeling planetary and stellar magnetospheres can be constructed. We present some explicit examples below. In each example where a magnetic field is partially open, we do not, as in § II, attempt to account for the dynamical processes that keep the field open. A magnetic field may open up because of a stellar wind. If there is rotation, this wind may be due to the centrifugal force.

We shall first consider an axisymmetric magnetosphere without rotation. Superpose Z_1 with a potential field made up of a dipole and a uniform magnetic field:

$$Z = Z_1 + \lambda_1 \frac{\sin^2\theta}{r} + \lambda_2 r^2 \sin^2\theta\ , \tag{30}$$

where λ_1 and λ_2 are free constants to control the magnitudes of the added potential fields. The cases of $\lambda_2 = 0$, $\lambda_1 \neq 0$ and $\lambda_1 = 0$, $\lambda_2 \neq 0$, adding only the dipole or the uniform potential fields, are shown in Figures 5*a* and 6*a* respectively; we refer to them as cases I and II. In each case, the added field is taken to be oriented relative to the equatorial current sheet such that the Lorentz force in the current sheet is everywhere outward. This Lorentz force can be balanced everywhere by the weight of a surface mass m distributed over the equatorial current sheet given by

$$\frac{GM_0 m}{r^2} = \frac{1}{c} J(B_\theta)_{\theta=\pi/2}\ , \tag{31}$$

where G is Newton's constant, M_0 the mass of the central star, J the surface electric current density, and c the speed of light. The surface mass m as a function of r, associated with cases I and II, takes the forms

$$m = \frac{\lambda_1 a^2}{\pi G M_0} \frac{1}{r^3} \left(1 - \frac{a^2}{r^2}\right)^{1/2}, \quad \text{case I}\ , \tag{32}$$

$$m = \frac{2\lambda_2 a^2}{\pi G M_0} \left(1 - \frac{a^2}{r^2}\right)^{1/2}, \quad \text{case II}\ , \tag{33}$$

and their profiles are shown in Figure 7. The magnetic configuration in case I may be interpreted to be due to an equatorial accretion disk intruding into a partially open stellar magnetic field. The accretion disk has its own magnetic field which is not connected to the central star. It is the tension force of this bow-shaped magnetic field that supports the weight of the ac-

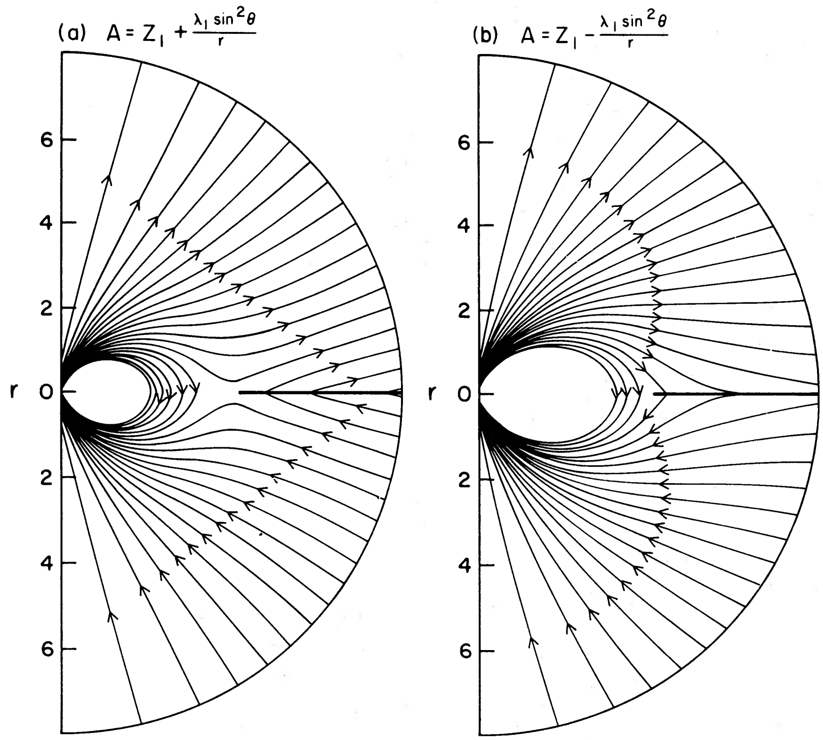

FIG. 5.—Lines of force in the (r, θ)-plane obtained by surperposing the Z_1-field with dipole potential fields of opposite signs, with $a = 4.0$ and $\lambda_1 > 0$. (*a*) Case I. (*b*) Case III.

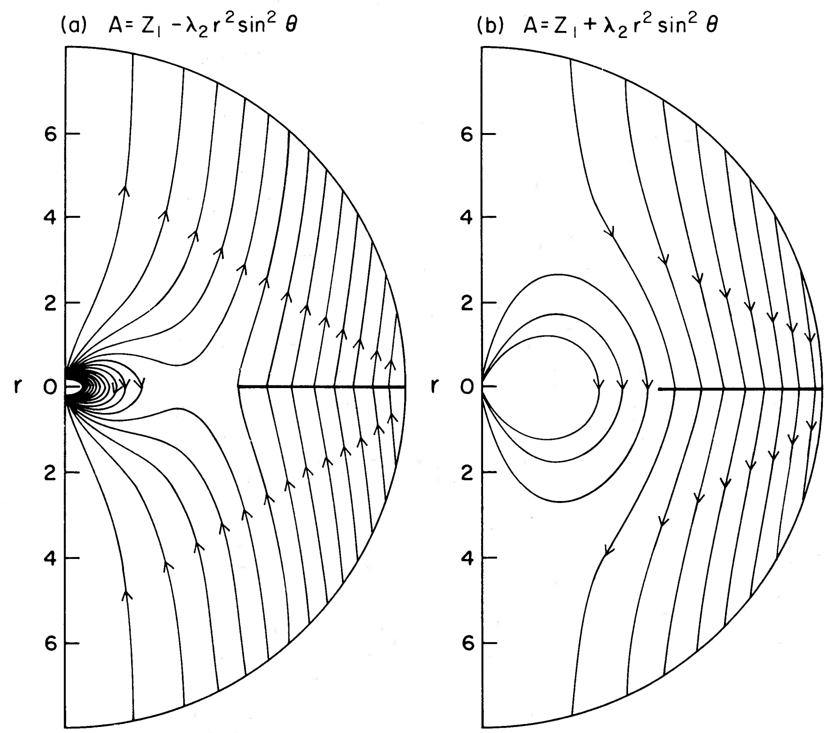

FIG. 6.—Lines of force in the (r, θ)-plane obtained by superposing the Z_1-field with uniform fields of opposite signs, with $a = 4.0$ and $\lambda_2 > 0$. (*a*) Case II. (*b*) Case IV.

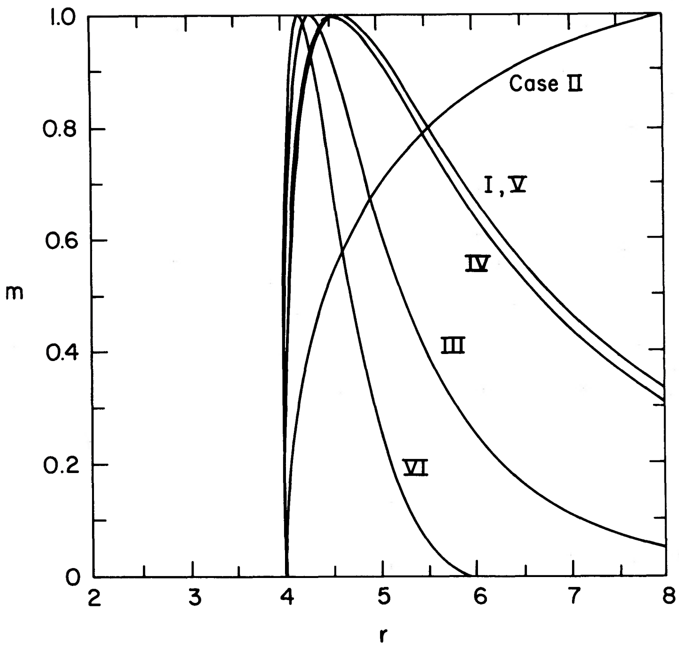

FIG. 7.—Profiles of the surface mass density $m(r)$ of the equatorial disk in the solutions displayed in Figs. 5, 6, 8, and 9. The profiles have been suitably normalized to fit onto the same graph. In each case, $a = 4.0$; for case VI, $a' = 6.0$.

cretion disk material and prevents the disk from falling into the star. A similar support suspends quiescent prominences in the solar corona (Kippenhahn and Schlüter 1957). The magnetosphere in case I is isolated in space so that, at large distances, the lines of force are radially open and the magnetic intensity falls to zero. With the particular value of λ_1 used to obtain Figure 5*a*, part of the open radial magnetic flux originates from the accretion disk. If the parameter λ_1 is set sufficiently large, the weight of the accretion disk is increased linearly and the entire open radial magnetic flux comes from the accretion disk. In this case, the lines of force originating from the central star all close back on the star. We omit presenting this case graphically. Case II has an external uniform magnetic field aligned parallel to the magnetic polar axis of the central star. We can imagine the uniform field to be embedded in the interstellar medium or to approximate the field contributed by a companion star located far from the central star. In case II, the accretion disk is trapped in the largely uniform external magnetic field, and the open part of the stellar magnetic field is not radial at large distances but bends over to open to infinity as part of the uniform field. Note that the electric current J is the same in both cases and the difference in the mass distributions is due to the added potential fields. In case I, the added dipole potential field drops rapidly with distance and m decreases with distance as $1/r^3$. In case II, the uniform field allows for a uniform mass distribution at large distances, where the Lorentz force decreases in direct proportion to the decreasing gravitational force.

Suppose the matnetosphere is rotating uniformly, with angular speed ω. The centrifugal force dominates at distances larger than r_c given by

$$r_c = \left(\frac{GM_0}{\omega^2}\right)^{1/3} . \tag{34}$$

If we reverse the signs of the added potential fields in Figures 5*a* and 6*a* without reversing the electric current in the sheet, the Lorentz force in the electric current sheet would be reversed, as shown in Figures 5*b* and 6*b*. Let us refer to these as cases III and IV respectively. If we set the inner radius of the current sheet to be larger than r_c, the inward Lorentz force can be balanced by the outward, centrifugally dominated, body force of a mass distribution given by

$$\left(\omega^2 r - \frac{GM_0}{r^2}\right) m = \frac{1}{c} J(B_\theta)_{\theta=\pi/2} \cdot \tag{35}$$

The mass distribution has the explicit forms

$$m = \frac{\lambda_1 a^2}{\pi\omega^2} \frac{1}{r^3} \left(1 - \frac{a^2}{r^2}\right)^{1/2} (r^3 - r_c^3)^{-1}, \text{ case III} , \tag{36}$$

$$m = \frac{2\lambda_2 a^2}{\pi\omega^2} \left(1 - \frac{a^2}{r^2}\right)^{1/2} (r^3 - r_c^3)^{-1} , \text{ case IV} , \tag{37}$$

with the profiles shown in Figure 7. It is interesting to note that reversing the added potential fields to obtain cases III and IV leads to the following features in field topologies. In case III, the system is isolated in space. Cool dense material in the rotating magnetosphere collects in the equator in the form of a disk to be held against being thrown out by the centrifugal force. The disk is held by magnetic field lines that are all anchored to the central star, as shown in Figure 5*b*. In case IV, the external uniform field has two important effects. It provides a buffer to hold the disk against the centrifugal force so that the magnetic field connected to the disk need not be anchored to the central star. Since the external uniform field is now antiparallel to the stellar magnetic axis, the stellar field must be everywhere closed. In both cases, the Lorentz force in the current sheet decreases with radial distance, whereas the centrifugal force increases linearly with distance. Hence, the mass that must be supported by the Lorentz force is a rapidly decreasing function of distance.

We go to case V, which is an example for a disk intruding into the region $r < r_c$, where gravity dominates over the centrifugal force. Consider the potential magnetic field shown in Figure 8*a*. This potential magnetic field is a combination of the dipole and uniform fields with the constants λ_1 and λ_2 chosen so that $\lambda_1/\lambda_2 = 2r_c^3$ so as to locate the equatorial X-type neutral point at $r = r_c$. By further choosing the right sign of the net added potential magnetic field, we obtain the field shown in Figure 8*b*, where a Lorentz force is found in the current sheet acting outward in $r < r_c$ and inward in $r > r_c$. This Lorentz force can be balanced by oppositely directed body forces in the two regions, as described by equation (35). It is fortuitious that this Lorentz force can support a mass distribution which has the same profile as that obtaining in case I, as shown in Figure 7.

The following feature of the above constructions is noteworthy. In order to balance gravitational and centrifugal forces in the equatorial plane, we require the Lorentz force to be strictly radial there. This requirement can be met so long as the added potential magnetic field has no radial field component in the equatorial plane; and there is the interesting possibility of adding a potential magnetic field having this property but varying with three dimensions. The result is a three-dimensional model of a stellar magnetosphere. The problem is simplest without rotation. For example, the potential function

$$\Phi = C_1 \frac{1}{r^2} P_1(\cos\theta) + C_2 \frac{1}{r^4} P_3^2(\cos\theta) \sin 2\phi , \tag{38}$$

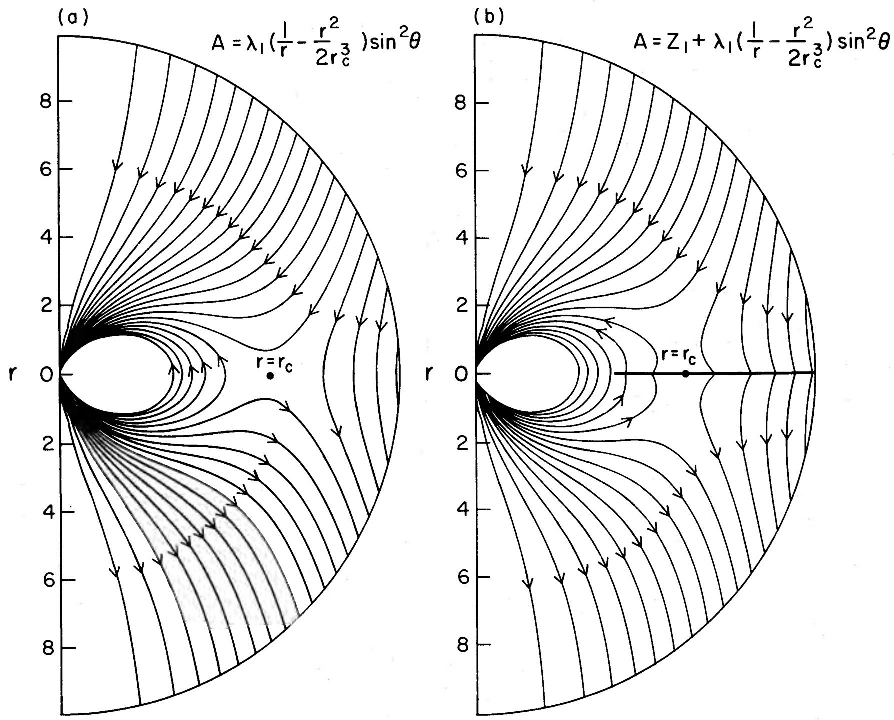

FIG. 8.—Lines of force in the (r, θ)-plane of the potential magnetic field made up of a dipole field aligned parallel to a uniform field and the superposition of this potential field with the Z_1-field. The neutral point of the potential field is located at $r = r_c$, where the centrifugal force in the disk balances the gravitational force of the central star. (*b*) Case V.

with constant coefficients C_1 and C_2, describes a three-dimensional field which has no radial component in the equatorial plane. By picking C_1 sufficiently larger than C_2, the component B_θ, which goes to determine the Lorentz force in the current sheet, is everywhere of a fixed sign. Then, adding this potential field to a current sheet solution allow us to construct a three-dimensional model in which the Lorentz force supports the weight of an equatorial mass sheet that varies with r and ϕ. We leave the interested reader to explore the infinite varieties of models that can be generated in this manner. Note that the stream-function representation cannot be extended to the three-dimensional potential magnetic field.

A magnetosphere rotating uniformly everywhere presents a serious problem of having distant parts of its structure moving at speeds that can exceed the speed of light. Classical description must break down. In particular, the fluid theory in the nonrelativistic limit fails to apply. It should be pointed out, however, that within the classical description adopted here, the solutions displayed in Figures 5–8 are formally acceptable. In these examples where the mass points in the far reaches of the rotating current sheet are moving at enormous speeds, the mass decreases with radial distance so rapidly that the rotational energy decreases to zero and is relatively unimportant in the far region. Accounting for a proper outer boundary, where relativistic effects are important, lies beyond the scope of this paper, but there are two ways to avoid the unbounded high corotational velocities at infinity. This undesired property can be removed by taking the rate of rotation to be a function of r (Vasyliunas 1983). The point to note is that equation (35) determines the product $m\omega^2$ for large r. A mass distribution modified from that of uniform rotation in this asymptotic region with a compensating reduced rate of rotation that vanishes at infinity removes the unbounded corotational velocities. Another procedure is to truncate the mass distribution at an outer radius beyond which the mass density is zero. To illustrate this possibility, consider the superposition

$$Z = Z_1(a) + Z_1(a') , \tag{39}$$

where the two terms in Z_1 have different values for the parameter characterizing the inner radii of their respective equatorial current sheet. The combined field, case VI, is shown in Figure 9, with $a' > a > r_c$, that is, the current sheet of $Z_1(a')$ has a hole bigger than the current sheet of $Z_1(a)$. The two current sheets overlap in the region $r > a'$, where the combined current sheet is stress-free with zero mass density. In the region $a' > r > a$ of the current sheet of $Z_1(a)$, which does not overlap, a Lorentz force acts radially inward to support a mass density given by

$$m = \frac{a}{\pi\omega^2}\frac{1}{r^4}\left(1 - \frac{a^2}{r^2}\right)^{1/2} \times\left[\left(\frac{a'2}{r^2} - 1\right) - \frac{a'2}{r^2}\cot^{-1}\left(\frac{a'^2}{r^2} - 1\right)^{-1/2}\right]\left(1 - \frac{r_c^3}{r^3}\right)^{-1} , \tag{40}$$

defined for the range $a < r < a'$, with the profile shown in Figure 7.

IV. DISCUSSION

One way of regarding the set of potential magnetic fields generated by the stream functions Z_n is to think of them as a generalization of the set of axisymmetric spherical harmonic potential fields of the same index n. Taking the solution Z_n, of a given n, to extend over all space, it reduces to the corresponding classical spherical harmonic potential function in the neighborhood of the origin, So, the function Z_n represents a modification of the corresponding spherical harmonic poten-

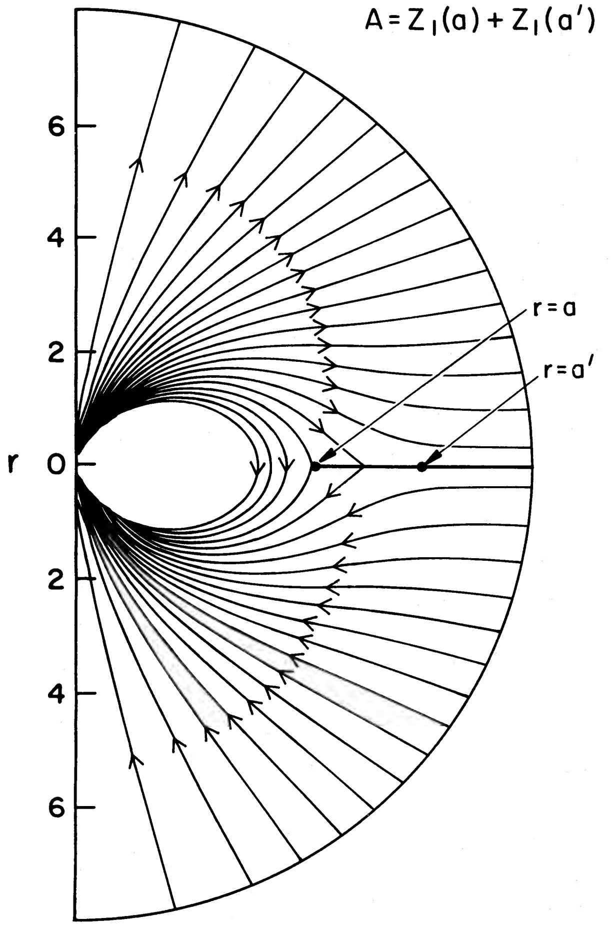

FIG. 9.—Lines of force in the (r, θ)-plane obtained by superposing the stream functions $Z_1(a)$ and $Z_1(a')$, with $a' > a > r_c$. The surface mass is loaded only in the region $a < r < a'$, with no mass in the region $a' < r_c$. Case VI.

tial function to allow for the presence of an infinite equatorial current sheet which opens up a part of the magnetic flux to infinity. The parameter a is the radius of the inner circular boundary of the infinite equatorial current sheet and is free to be adjusted to control the amount of open magnetic flux. This generalization is restricted to fields that are axisymmetric and symmetric about the equator. This restriction on geometry is required in order to locate the current sheet on the equator, a simplification which facilitated the direct construction of the solutions in §§ II and III. It seems clear that the other spherical harmonic potential functions, lacking the symmetry assumed here, also have similar generalizations. To construct these other generalizations requires treating the current sheet as curved surfaces. For example, a north-south asymmetry would require a current sheet which is warped at near distances but, at large distances, may approach the form of an equatorial sheet perpendicular to the net dipole moment of the star. Treating this free-boundary problem is a challenging further development of the work reported here.

In this paper, we have a limited but versatile set of current sheet solutions. The versatility comes from the availability of the principle of superposition, which allows us to generate a large variety of solutions. The superposition principle also allows us to extend the class of solutions to include those that have stressed current sheets and, in a limited way, those with variations in three dimensions. We have chosen only the simplest examples for presentation in §§ II and III, namely, the $n = (1, 3)$ solutions. As a simple physical illustration, let us estimate the total mass of the magnetodisk in the Jovian magnetosphere, based on the magnetic configuration in Figure 5*b* and using certain macroscopic parameters of the disk electric current obtained from satellite observations by Connerney, Acuña, and Ness (1981). Jupiter has an equatorial radius of $R_J = 7.14 \times 10^9$ cm, total mass $M_0 = 1.9 \times 10^{30}$ g, a magnetic dipole strength of $D = 4.2$ G R_J^{-3}, and a rotation rate of $\omega = 1.76 \times 10^{-4}$ rad s^{-1}. This fixes the point of balance between centrifugal and gravitational forces at $r_c = 2.24 R_J$. Connerney *et al.* gave the inner radius of the equatorial current sheet to be $a \approx 5R_J$, with 60% of the total electric current lying inside $r = 20R_J$. The current sheet therefore lies in the region $r > r_c$, where the centrifugal force is stronger than the planet's gravitational force. The magnetic field in Figure 5*b* is generated by

$$Z = \gamma_1\left(Z_1 - \frac{\lambda_1 \sin^2\theta}{r}\right), \quad (41)$$

where we insert a constant amplitude γ_1. To fix the two constants γ_1 and λ_1, we fit the dipole strength at $r = 0$ given by equation (41) to the Jovian value of 4.2 G R_J^{-3} and demand that the field strength at $r = 30R_J$ on the equator is twice that due to the internal source of Jupiter, as reported by Connerney *et al.* Having thus determined the magnetic field, we find that the electric current density peaks at $r = 6.1R_J$ and 60% of the total current lies inside of $r = 16R_J$. The disk then has a mass density that peaks at $r = 5.4R_J$ and declines rapidly as r^{-6} at large distances. The total mass is $\sim 4.4 \times 10^{12}$ g, of which 60% lies inside $r = 5.5R_J$. These are crude estimates made with the convenience of the closed-form solution for the magnetic field given in equation (41). A particular limitation of this solution is that the magnetic field of Jupiter is represented simply by a dipole source at the origin. In a more refined calculation, we should allow for the fact that the presence of the electric current sheet so close to the planet's surface, with the inner radius of the current sheet located at $5R_J$, and the shielding effect of the planet's ionized atmosphere can induce an effective multipole component to the planet's contribution to the external magnetic field. The net field is than likely to be a superposition of the various W_n and Z_n solutions with $n = 1, 3, 5, \ldots$ The distribution of the electric current in the sheet and the normal field at the sheet in the near zone is then different from that of the simple case in equation (41). The equilibrium mass density distribution in the near zone is accordingly modified. In the far zone, only the dipole term in the W_n expansion dominates, while the Z_n take on the asymptotic form of an equatorial current sheet separating opposite radial magnetic fields. The latter has a current density that declines as r^{-2}. The mass distribution in the far region is therefore basically unchanged from that given by equation (41), namely, one that declines as r^{-6}. It is instructive to relate our magnetospheric models to those reported by Gleeson and Axford (1976) and Hill and Carbary (1978). In these other models, the starting point of the construction has been the free prescription of the distribution of the electric current density in the equatorial current sheet. The magnetic field due to this prescribed electric current density is then superposed with the dipole potential field of Jupiter to give the magnetic field everywhere. Gleeson and Axford considered two explicit examples: the $K_1 = 1$,

$K_2 = 0$ and $K_1 = 0$, $K_2 = 1$ solutions, in their notation. The former has a current sheet whose prescribed electric current density declines like r^{-2}. This current distribution is consistent with an external magnetic field in the far region which is radial, of opposite signs on either sides of the current sheet, and declining like r^{-2}. The solution can, in principle, be expressed as a linear combination of our W_n and Z_n solutions. The second solution of Gleeson and Axford has a current sheet whose electric current density in the far region declines faster than r^{-2}. This current distribution is not consistent with an open magnetic field in the far region. The closed geometry of the magnetic field in the far region requires the appearance of magnetic neutral points in the current sheet. Negative mass density can arise from changes in the sign of the Lorentz force across the neutral points; and from this, it was suggested that such a current distribution requires a breakdown of corotation and the presence of a magnetospheric wind, as considered in the model of Hill and Carbary (1978). The second model of Gleeson and Axford is of course not included in the set of linear combinations of the W_n and Z_n solutions. The Z_n solutions have been constructed with an explicit demand that all magnetic field lines are open in the far region.

It is clear that a rich variety of idealized, analytic models has become available for illustrating the global magnetic configurations of the Sun and magnetospheres of planets and stars with rotating or nonrotating equatorial disks. We should also point out that the availability of the explicit solutions derived in this paper presents an opportunity to study linear hydromagnetic stability of these equilibrium states by the classical techniques of perturbation expansion (Wu 1986).

This work orginated from a conversation with Art Hundhausen, who asked if partially open potential magnetic fields could be found in closed form. I thank him and Tom Bogdan for discussions.

APPENDIX A

MATHEMATICAL FORMULAE USED IN THE TEXT

We first note that the oblate spheroidal coordinates ξ and η have the range $0 < \xi < \infty$, $-1 < \eta < 1$, generating elliptic and hyperbolic surfaces of revolution respectively. There are four degenerate surfaces. The surface $\xi = 0$ is the degenerate elliptic surface of revolution, forming a disk of radius a centered at the origin. The surface $\eta = 1$ and $\eta = -1$ are the positive and negative z-axes respectively. Finally, the surface $\eta = 0$ is the equatorial plane with a circular hole of radius a centered at the origin. It follows that ξ is continuous everywhere and η is continuous in $r > a$ but is discontinuous with a change of sign across $\theta = \pi/2$ in $r < a$. The degenerate surfaces $\xi = 0$ and $\eta = 0$ require careful evaluation of the derivatives taken on them. These derivatives are either zero and continuous or else discontinuous with a change of signs, across the equator. In the following equations, we give the various derivatives evaluated by approaching the equator from $\theta < \pi/2$.

i) *Values on* $\theta = \pi/2, r < a$:

$$\xi = \frac{\partial \xi}{\partial r} = \frac{\partial \eta}{\partial \theta} = 0 , \tag{A1}$$

$$\eta = \left(1 - \frac{r^2}{a^2}\right)^{1/2} , \tag{A2}$$

$$\frac{\partial \xi}{\partial \theta} = -\frac{r}{a}\left(1 - \frac{r^2}{a^2}\right)^{-1/2} , \tag{A3}$$

$$\frac{\partial u}{\partial \theta} = v = \frac{\partial v}{\partial r} = 0 , \tag{A4}$$

$$u = \left(\frac{a^2}{r^2} - 1\right)^{1/2} , \tag{A5}$$

$$\frac{\partial v}{\partial \theta} = -\frac{a}{r}\left(\frac{a^2}{r^2} - 1\right)^{-1/2} . \tag{A6}$$

ii) *Values on* $\theta = \pi/2, r > a$:

$$\frac{\partial \xi}{\partial \theta} = \eta = \frac{\partial \eta}{\partial r} = 0 , \tag{A7}$$

$$\xi = \left(\frac{r^2}{a^2} - 1\right)^{1/2} , \tag{A8}$$

$$\frac{\partial \eta}{\partial \theta} = -\frac{r}{a}\left(\frac{r^2}{a^2} - 1\right)^{-1/2} , \tag{A9}$$

$$u = \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = 0 , \tag{A10}$$

$$v = \left(1 - \frac{a^2}{r^2}\right)^{1/2} , \tag{A11}$$

$$\frac{\partial u}{\partial \theta} = -\frac{a}{r}\left(1 - \frac{a^2}{r^2}\right)^{-1/2} . \tag{A12}$$

APPENDIX B

CONSTRUCTION OF MAGNETIC FIELDS USING POTENTIAL FUNCTION

In the oblate spheroidal coordinates, Laplace equation (3) takes the form

$$\frac{\partial}{\partial \xi}(\xi^2 + 1)\frac{\partial \Phi}{\partial \xi} + \frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial \Phi}{\partial \eta} + \frac{\xi^2 + \eta^2}{(\xi^2 + 1)(1 - \eta^2)}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 . \tag{B1}$$

With axisymmetry, we have the solutions

$$\Phi_n = P_n(\eta)Q_n(i\xi) , \tag{B2}$$

where n is an integer (Morse and Feshbach 1953). These are the potential functions corresponding to the stream functions A_n given by equation (14). Carrying out the process of inversion with respect to the sphere of radius a centered at the origin and rendering the equatorial current sheet stress-free, it is straightforward to develop from Φ_n, with n odd, a set of potential functions describing potential magnetic fields with equatorial, stress-free, current sheets, in the same manner as was done to obtain the stream functions Z_n in § II. Let us denote these potential functions by Ψ_n. It is important to note that if a potential magnetic field $\boldsymbol{B}$ has potential Φ and stream function A, the Kelvin transformation applied to Φ and A separately, in the two different representations, may not yield the same transformed magnetic field. The potential functions Ψ_n have been obtained by first performing the Kelvin transformation on Φ_n and, for the same n, they give magnetic fields which are quite distinct from the magnetic fields generated by the stream functions Z_n, which are derived from the Kelvin transformation on the stream functions A_n. It can be shown that, for each given n, the magnetic field having the stream function Z_n can be expressed linearly in terms of those given by the potential functions Ψ_k with $k = 1, 2, 3, \ldots$. A similar linear retationship exists between the magnetic field with the potential function Ψ_n and the magnetic fields having stream functions Z_k with $k = 1, 2, 3, \ldots$. For the interested reader, we give the explicit forms of Z_n and Ψ_n for $n = 1$ and 3:

$$Z_1 = r(1 - v^2)\left[(1 + u^2)\tan^{-1}\frac{1}{u} - u\right] - \frac{\pi a^2}{2}\frac{\sin^2\theta}{r} + 2a\eta , \tag{B3}$$

$$Z_3 = \frac{3}{4}r(1 - v^2)(5v^2 - 1)\left[-3(1 + u^2)(5u^2 + 1)\tan^{-1}\frac{1}{u} + 15u^3 + 13u\right] + \frac{45\pi a^4}{8}\frac{\sin^2\theta(5\cos^2\theta - 1)}{r^3} + \frac{9\pi a^2}{2}\frac{\sin^2\theta}{r} + 12a\eta , \tag{B4}$$

$$\Psi_1 = \frac{a}{r}v\left(u\tan^{-1}\frac{1}{u} - 1\right) - \tan^{-1}\frac{1}{\xi} - \frac{\pi a^2}{2}\frac{\cos\theta}{r^2} , \tag{B5}$$

$$\Psi_3 = \frac{a}{r}(5v^3 - 3v)\left[-\frac{1}{2}u(5u^2 + 3)\tan^{-1}\frac{1}{u} + \frac{5}{2}u^2 + \frac{2}{3}\right] + \frac{5\pi a^4}{4}\frac{5\cos^3\theta - 3\cos\theta}{r^4} + \frac{3\pi a^2}{2}\frac{\cos\theta}{r^2} - 2\tan^{-1}\frac{1}{\xi} . \tag{B6}$$

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